

SEQUENTIALITY RESTRICTIONS IN SPECIAL RELATIVITY

MARK I. HEILIGMAN

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ABSTRACT. Observers in different inertial frames can see a set of spacelike separated events as occurring in different orders. Various restrictions are studied on the possible orderings of events that can be observed. In $1 + 1$ -dimensional spacetime $\{(123), (231), (312)\}$ is a disallowed set of permutations. In $3 + 1$ -dimensional spacetime, any four different permutations on the ordering of n events can be seen by four different observers, and there is a set of four events such that any of the $4! = 24$ possible orderings can be observed in some inertial reference frame. A more complicated problem is that of five observers and five events, where of the 7,940,751 choices of five distinct elements from \mathfrak{S}_5 (containing the identity), all but at most one set of permutations can be realized, and it is shown that this remaining case is impossible. For six events and five observers, it is shown that there are at least 7970 cases that are unrealizable, of which at least 294 do not come from the forbidden configuration of five events.

INTRODUCTION

One of the well known “paradoxes” of special relativity is that two observers who are moving in different inertial reference frames will disagree about whether two events in space-time are simultaneous. Consequently, they can also disagree about the sequential order in which events in space-time occur. The purpose of this note is to begin investigating whether there are restrictions on the relative orderings of these events.

Generally for n events in spacetime, an observer will see them occurring in some order, which can be viewed as an element of \mathfrak{S}_n , the symmetric group of permutations of the set $\{1, 2, \dots, n\}$. Each inertial frame of reference will assign an element $\pi \in \mathfrak{S}_n$ to the observed sequence of events. The basic question is whether there are restrictions on which subsets of \mathfrak{S}_n can occur for multiple observers of the same n events.

In all that follows, the speed of light c will be taken to be 1, and all velocities will be taken as the fraction of the speed of light.

Key words and phrases. Minkowski Space, Hyperplane Arrangements.

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Historical Note.

A number of these ideas were discussed in Professor Richard Stanley's paper [S]. In that paper, he mentions a mathematician from NSA whose name he can't recall who made the connection between event orderings in Minkowski space with hyperplane arrangements. The author of the present paper is that mathematician, who at the time of the discussion was working at NSA. The current author truly appreciates the reference from [S], and was deeply gratified in the course of checking out the background references for this paper to find that Professor Stanley found the question to be of sufficient interest to write on the topic.

THE 1-DIMENSIONAL CASE

Consider the situation of events on a line, so that their space-time coordinates consist only of an x -coordinate and a time. For an observer in an inertial frame, the n events have coordinates (x_i, t_i) for $i = 1, 2, \dots, n$. Any other inertial frame is related to this rest frame by a characteristic velocity v , in the range $-1 < v < 1$.

For another observer in an inertial frame moving at velocity v with respect to the initial frame, the i -th event will have coordinates (x'_i, t'_i) where

$$\begin{aligned} x'_i &= \gamma (x_i - v t_i) \\ t'_i &= \gamma (t_i - v x_i) \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

is the Lorentz factor.

Now suppose that $\sigma \in \mathfrak{S}_n$ is a permutation, and set $t_i = i$ and $x_i = 2(i - \sigma(i))$ and let $v = \frac{1}{2}$. Then $t'_i = \gamma \sigma(i)$, so while an observer in the rest frame sees the events in order $1, 2, \dots, n$, an observer in an inertial frame moving with velocity $v = \frac{1}{2}$ with respect to the rest frame sees the events in order $\sigma(1), \sigma(2), \dots, \sigma(n)$. This shows that for any pair of permutations in \mathfrak{S}_n there are n events and a pair of inertial frames such that each observer sees the events with their respective permutations on the order of occurrence.

On the other hand, consider the case of three events with three observers and whether it is possible for one observer to see the events in order $(1, 2, 3)$, the second observer to see the events in order $(2, 3, 1)$ and the third observer to see the events in order $(3, 1, 2)$. From the perspective of at any observer who numbers the events as $(1, 2, 3)$, the other two observers number the events as $(2, 3, 1)$ and $(3, 1, 2)$. Also, for one of the observers, which will be the rest frame for the purposes of this discussion, the other two reference frames are moving with velocities $v^{(1)}$ and $v^{(2)}$ that are both positive. If the events have coordinates (x_i, t_i) for $i = 1, 2, 3$ in the rest frame, then $t_1 < t_2 < t_3$, while in the other two frames

$$\begin{aligned} t_2^{(1)} &= \gamma^{(1)}(t_2 - v^{(1)} x_2) < t_3^{(1)} = \gamma^{(1)}(t_3 - v^{(1)} x_3) < t_1^{(1)} = \gamma^{(1)}(t_1 - v^{(1)} x_1) \\ t_3^{(2)} &= \gamma^{(2)}(t_3 - v^{(2)} x_3) < t_1^{(2)} = \gamma^{(2)}(t_1 - v^{(2)} x_1) < t_2^{(2)} = \gamma^{(2)}(t_2 - v^{(2)} x_2) \end{aligned}$$

or simply extracting out some relevant inequalities

$$\begin{aligned} t_2 - v^{(1)} x_2 &< t_1 - v^{(1)} x_1 \\ t_3 - v^{(1)} x_3 &< t_1 - v^{(1)} x_1 \\ t_3 - v^{(2)} x_3 &< t_1 - v^{(2)} x_1 \\ t_3 - v^{(2)} x_3 &< t_2 - v^{(2)} x_2. \end{aligned}$$

Now adding the inequalaties $t_1 < t_2 < t_3$ appropriately gives

$$\begin{aligned} t_1 + t_2 - v^{(1)} x_2 &< t_2 + t_1 - v^{(1)} x_1 \\ t_1 + t_3 - v^{(1)} x_3 &< t_3 + t_1 - v^{(1)} x_1 \\ t_1 + t_3 - v^{(2)} x_3 &< t_3 + t_1 - v^{(2)} x_1 \\ t_2 + t_3 - v^{(2)} x_3 &< t_3 + t_2 - v^{(2)} x_2. \end{aligned}$$

so that

$$\begin{aligned} v^{(1)} x_1 &< v^{(1)} x_2 \\ v^{(1)} x_1 &< v^{(1)} x_3 \\ v^{(2)} x_1 &< v^{(2)} x_3 \\ v^{(2)} x_2 &< v^{(2)} x_3 \end{aligned}$$

and therefore $x_1 < x_2 < x_3$ since $v^{(1)} > 0$ and $v^{(2)} > 0$.

Now notice that $\gamma^{(1)}(t_2 - v^{(1)} x_2) < \gamma^{(1)}(t_3 - v^{(1)} x_3)$ and $\gamma^{(2)}(t_3 - v^{(2)} x_3) < \gamma^{(2)}(t_2 - v^{(2)} x_2)$ implies

$$(t_2 - v^{(1)} x_2) + (t_3 - v^{(2)} x_3) < (t_3 - v^{(1)} x_3) + (t_2 - v^{(2)} x_2)$$

and therefore

$$(v^{(1)} - v^{(2)})(x_3 - x_2) = v^{(1)} x_3 + v^{(2)} x_2 - v^{(1)} x_2 - v^{(2)} x_3 < 0$$

which implies $v^{(1)} - v^{(2)} < 0$ since $x_3 - x_2 > 0$.

On the other hand, $\gamma^{(1)}(t_2 - v^{(1)} x_2) < \gamma^{(1)}(t_1 - v^{(1)} x_1)$ and $\gamma^{(2)}(t_1 - v^{(2)} x_1) < \gamma^{(2)}(t_2 - v^{(2)} x_2)$ implies

$$(t_2 - v^{(1)} x_2) + (t_1 - v^{(2)} x_1) < (t_1 - v^{(1)} x_1) + (t_2 - v^{(2)} x_2)$$

and therefore

$$(v^{(2)} - v^{(1)})(x_2 - x_1) = v^{(2)} x_2 + v^{(1)} x_1 - v^{(1)} x_2 - v^{(2)} x_1 < 0$$

which implies $v^{(2)} - v^{(1)} < 0$ since $x_2 - x_1 > 0$.

This gives a contradiction, thereby showing that the triple of permutations $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ is not realizable by three observers in $1 + 1$ relativistic spacetime.

It might be guessed that this example provides an essential obstruction, and indeed any set of three permutations that contains a subset of events that follows

the pattern $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ is unrealizable. However, it is not the case that all such unrealizable sets of three permutations contain a subset of this type.

Notice that the essential issue is that as the velocity increases, if the order of a pair of events flips, then it can't flip back. For example, the three permutations on six events $\{(1, 2, 3, 4, 5, 6), (2, 1, 3, 4, 6, 5), (1, 2, 4, 3, 6, 5)\}$ is a set of permutations that cannot occur in 1+1 Minkowski spacetime because regardless of how they are ordered, one of the pairs $\{1, 2\}$ or $\{3, 4\}$ or $\{5, 6\}$ flips its order and then flips back. This example is interesting because there is no subset of the events that follows the pattern $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

If $\pi \in \mathfrak{S}_n$, a reversal within π is a pair (i, j) with $i < j$ such that $\pi(j) < \pi(i)$. Let $\rho(\pi)$ denote the set of reversals, i.e.

$$\rho(\pi) = \{(i, j) \mid 1 \leq i < j \leq n \text{ \& } \pi(j) < \pi(i)\}.$$

If $\pi_1, \pi_2 \in \mathfrak{S}_n$, write $\pi_1 < \pi_2$ if $\rho(\pi_1) \subset \rho(\pi_2)$. This gives a partial order on \mathfrak{S}_n . With respect to this ordering, the minimal element is the identity permutation $\pi^{(0)} = (1, 2, \dots, n-1, n)$ and the maximal element is the reverse of the identity $(n, n-1, \dots, 2, 1)$.

Without any loss of generality, in considering whether a set of permutations is realizable, it may be assumed that one of the permutations is the identity. It is easily seen that a necessary condition for a set $\{\pi_1, \pi_2, \dots, \pi_k\}$ of permutations to be realizable is that they can be put in order with $\pi^{(0)} = \pi_1 < \pi_2 < \dots < \pi_k$. This was pointed out in [S].

It might be hoped that this criterion, namely that there is an ordering of permutations, would also be a sufficient condition for a set of permutations to be realizable. Alas, as pointed out in [S], the paper [GP] gives a counterexample of a set of five permutations $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ with π_0 the identity and with $\rho(\pi_1) \subset \rho(\pi_2) \subset \rho(\pi_3) \subset \rho(\pi_4)$ that is not realizable.

Nevertheless, an interesting question and perhaps one that is easier, would be to consider a set of three observers in 1 + 1-dimensional Minkowski space, and assume that one of the observers in the rest frame sees the events in order $\pi^{(0)}$ and that the other two observers see the events in order π_1 and π_2 , and that $\pi_1 < \pi_2$. Then is it possible to find an arrangement of n events in spacetime and a pair of velocities for observers 1 and 2 that realize these two permutations? The example from [GP] is not a counterexample to this question, since it involves more than three observers.

THE GENERAL CASE

Consider the situation of $m + 1$ observers of n events. The i -th observer sees the k -th event E_k as having space-time coordinates $(t^{(i)}(E_k), \tilde{\mathbf{w}}^{(i)}(E_k))$. Assume that the observers are numbered $i = 0, 1, \dots, m$ and that the events are numbered $j = 1, \dots, n$. If $\tilde{\mathbf{v}}^{(i)}$ is the velocity of observer i as seen by observer 0, then the Lorentz transformations give

$$t^{(i)}(E_k) = \gamma^{(i)} (t^{(0)}(E_k) - \tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{w}}^{(0)}(E_k))$$

where

$$\gamma^{(i)} = \gamma(\tilde{\mathbf{v}}^{(i)}) = \frac{1}{\sqrt{1 - \tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{v}}^{(i)}}}$$

is the Lorentz contraction factor. In all of this, $\tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{v}}^{(i)} < 1$, with the speed of the i -th observer being $\sqrt{\tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{v}}^{(i)}}$ as a fraction of the speed of light. Note also that $\gamma^{(i)} > 0$ for all i .

If $t^{(0)}(E_j) < t^{(0)}(E_k)$, but $t^{(0)}(E_k) - \tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{w}}^{(0)}(E_k) < t^{(0)}(E_j) - \tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{w}}^{(0)}(E_j)$, then observer i sees event E_k as preceding event E_j while the observer in the rest frame sees event E_j as preceding event E_k . The equation $t^{(0)}(E_k) - t^{(0)}(E_j) = \tilde{\mathbf{v}}^{(i)} \cdot (\tilde{\mathbf{w}}^{(0)}(E_k) - \tilde{\mathbf{w}}^{(0)}(E_j))$ determines a hyperplane in the space of velocities, which separate observers into two sets depending on whether E_j is before or after E_k . If there are n points in spacetime, then there are $\binom{n}{2}$ such separating hyperplanes. Each region of such a hyperplane arrangement corresponds to a different set of time orderings of the events to the different observers, i.e. to different subsets of \mathfrak{S}_n .

To establish some basic notation, for an observer in reference frame j , the time of an event E is $t^{(j)}(E)$ while the spatial coordinates of event E are $\tilde{\mathbf{w}}^{(j)}(E)$. From the point of view of reference frame 0, the velocity of reference frame j is $\tilde{\mathbf{v}}^{(j)}$ and the corresponding Lorentz factor is

$$\gamma^{(j)} = \frac{1}{\sqrt{1 - \tilde{\mathbf{v}}^{(j)} \cdot \tilde{\mathbf{v}}^{(j)}}}$$

which is always positive. The Lorentz transformation then gives

$$t^{(j)}(E) = \gamma^{(j)} \left(t^{(0)}(E) - \tilde{\mathbf{v}}^{(j)} \cdot \tilde{\mathbf{w}}^{(0)}(E) \right).$$

Definition. If $t^{(j)}(E_{i_1}) < t^{(j)}(E_{i_2}) < \dots < t^{(j)}(E_{i_n})$, then the reference frame j , or the observer j , sees the events in order $(i_1, i_2, \dots, i_n) \in \mathfrak{S}_n$

Each observer sees the n events under consideration in some order that can be described by an element $\sigma \in \mathfrak{S}_n$. For the i -th observer, suppose that the order in which the events are observed is given by the permutation $\sigma_i \in \mathfrak{S}_n$, so that

$$t^{(i)}(E_{\sigma_i(1)}) < t^{(i)}(E_{\sigma_i(2)}) < \dots < t^{(i)}(E_{\sigma_i(n)})$$

which can be expressed as saying that if $j < k$ then $t^{(i)}(E_{\sigma_i(j)}) < t^{(i)}(E_{\sigma_i(k)})$. Without loss of generality, it may be assumed that the events are labeled so that observer 0 sees them in order $1, 2, \dots, n$ so that $t^{(0)}(E_1) < t^{(0)}(E_2) < \dots < t^{(0)}(E_n)$ which means that $\sigma_0 \in \mathfrak{S}_n$ is the identity permutation.

Definition. A set $Q = \{\pi_1, \dots, \pi_k\} \subset \mathfrak{S}_n$ of cardinality k is said to be *realizable* in $d + 1$ -dimensional Minkowski space if there is a set of n events in $d + 1$ -dimensional Minkowski space and a set of k inertial reference frames in $d + 1$ -dimensional Minkowski space such that observer i sees the n events in order π_i for $i = 1, \dots, k$.

It is important to realize that there is no preferred labeling of the events, so if the events are relabeled by $\sigma^{-1} \in \mathfrak{S}_n$, the set Q becomes $Q\sigma = \{\pi_1\sigma, \dots, \pi_k\sigma\}$.

Definition. If $Q \subset \mathfrak{S}_n$ and if $\sigma \in \mathfrak{S}_n$ then the set $Q\sigma$ is said to be equivalent to Q . The equivalent sets to Q that contain the identity permutation are of the form $Q\pi^{-1}$ as π ranges over the elements of Q . If Q is actually a subgroup of \mathfrak{S}_n then the only subset of \mathfrak{S}_n that contains the identity and is equivalent to Q is Q itself.

Since special relativity is invariant under time reversal, it is useful to consider the time reversal of a set of events. Let $\pi_r = (n, n-1, \dots, 2, 1)$ be the permutation that reverses the labeling of all the elements. Applying π_r on the left simply reverses the order of any permutation.

Definitions. If $Q \subset \mathfrak{S}_n$, its *time reversal* is the set $\pi_r Q \pi_r$. The set Q is *time reversal invariant* if $\pi_r Q \pi_r = Q$.

Counting invariant sets.

It is interesting to count the number of time reversal invariant sets of \mathfrak{S}_n of size k that contain the identity permutation π_0 .

The permutation π_r is an involution (i.e. $\pi_r^2 = 1$). Let C be the set of elements of \mathfrak{S}_n that are invariant under conjugation by π_r and let $c = |C|$ be the number of elements of \mathfrak{S}_n that are invariant under conjugation by π_r . The number of pairs of elements of \mathfrak{S}_n that are conjugates under π_r is then $c' = (n! - c)/2$ and $c' = |C'|$ where C' is the set of pairs of elements of \mathfrak{S}_n that are conjugate under π_r .

If $\pi_r Q \pi_r = Q$, then the elements of Q are either invariant under conjugation by π_r or come in pairs that are conjugates under π_r . Let i be the number of elements of Q that are invariant under conjugation by π_r , so $j = (k - i)/2$ is the number of conjugate pairs in Q . The number of ways of choosing i elements of C , where one of the elements is π_0 , the identity permutation is $\binom{c-1}{i-1}$ and the number of ways of choosing $j = (k - i)/2$ elements of C' is $\binom{c'}{j}$. Since $i = k - 2j$, the total number of time reversal invariant sets Q is

$$\sum_{j=0}^J \binom{c'}{j} \binom{c-1}{k-2j-1}$$

where $J = (k - 1)/2$ if k is odd, and $J = (k - 2)/2$ if k is even, and in either case $J = \lfloor \frac{k-1}{2} \rfloor$. Since $c' = (n! - c)/2$, all that remains is to find c .

From the group theory perspective, $C = C(\pi_r)$ is the centralizer of π_r , which is the subgroup of all permutations that commute with π_r , so that $c = |C(\pi_r)|$. The right cosets of $C(\pi_r)$ are in one-to-one correspondence with conjugacy classes of π_r in \mathfrak{S}_n . In turn, the conjugacy class of π_r in \mathfrak{S}_n consists of all permutations having the same cycle structure as π_r .

If n is even then π_r is a product of $\frac{n}{2}$ 2-cycles, i.e. $\pi_r = (1\ n)(2\ n-1) \cdots (\frac{n}{2}\ \frac{n}{2}+1)$, while if n is odd then π_r is a product of $\frac{n-1}{2}$ 2-cycles and a single 1-cycle, i.e. $\pi_r = (1\ n)(2\ n-1) \cdots (\frac{n-1}{2}\ \frac{n+3}{2})(\frac{n+1}{2})$.

To determine the number of ways that $\lfloor \frac{n}{2} \rfloor$ 2-cycles can be made, note that there are $\binom{n}{2}$ ways of picking the first 2-cycle, and then after that there are $\binom{n-2}{2}$ ways of picking the second 2-cycle, after which there are $\binom{n-4}{2}$ ways of picking the third 2-cycle, etc. for a total of

$$\binom{n}{2} \cdot \binom{n-2}{2} \cdot \binom{n-4}{2} \cdots \binom{n-2\lfloor \frac{n-2}{2} \rfloor}{2} = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}$$

ways of picking the needed numbers of 2-cycles. However, this needs to be divided by the number of ways that these two cycles can be ordered, since such a reordering gives rise to the same permutation in the end. Thus the total number of permutations having the same cycle structure as π_r is $n! / (\lfloor \frac{n}{2} \rfloor! 2^{\lfloor \frac{n}{2} \rfloor})$, which is the size of the conjugacy class of π_r in \mathfrak{S}_n . Therefore the size of the centralizer of π_r is

$$c = \lfloor \frac{n}{2} \rfloor! 2^{\lfloor \frac{n}{2} \rfloor}.$$

For $n = 5$, $c = 8$ and $c' = 56$, while for $n = 6$, $c = 48$ and $c' = 336$. For $k = 5$ observers $J = 2$ and therefore for $n = 5$, the number of time reversal invariant subsets of \mathfrak{S}_5 of size 5 is

$$\binom{48}{0}\binom{7}{4} + \binom{48}{1}\binom{7}{2} + \binom{48}{2}\binom{7}{0} = 2171$$

while for $n = 6$, the number of time reversal invariant subsets of \mathfrak{S}_6 of size 5 is

$$\binom{336}{0}\binom{47}{4} + \binom{336}{1}\binom{47}{2} + \binom{336}{2}\binom{47}{0} = 597861$$

Linearly dependent velocities.

Now suppose that there is a linear dependence between the velocities of all the observers, so that

$$\sum_{i=1}^m \alpha_i \tilde{\mathbf{v}}^{(i)} = 0$$

for some real numbers $\alpha_i \in \mathbb{R}$. (Note that $\tilde{\mathbf{v}}^{(0)} = 0$ so that α_0 is irrelevant in this linear relation.) Then

$$\begin{aligned} \sum_{i=1}^m \gamma^{(i)-1} \alpha_i t^{(i)}(E_k) &= \sum_{i=1}^m \alpha_i (t^{(0)}(E_k) - \tilde{\mathbf{v}}^{(i)} \cdot \tilde{\mathbf{w}}^{(0)}(E_k)) \\ &= \left(\sum_{i=1}^m \alpha_i \right) t^{(0)}(E_k) - \left(\sum_{i=1}^m \alpha_i \tilde{\mathbf{v}}^{(i)} \right) \cdot \tilde{\mathbf{w}}^{(0)}(E_k) \\ &= A t^{(0)}(E_k) \end{aligned}$$

It is important to assume now that $\alpha_i \neq 0$ for $i = 1, \dots, m$ and that $A \neq 0$, as well. These actually are quite mild assumptions, since the set of m velocities that will satisfy any set of strict sequentiality inequalities will be an open set, slight variations in the velocities will not affect the ordering of events for the various observers. These slight variations in the $\tilde{\mathbf{v}}^{(i)}$'s will be enough to insure that without any loss of generality, it should be possible to take all the α_i 's to be non-zero and at the same time also take their sum A to be nonzero.

With this assumption in hand, it is also clear that the α_i 's can all be multiplied by any nonzero real number, and the linear dependence of the velocities will remain. Without loss of generality, then, it can be assumed that the α_i 's are such that $A > 0$. Note that this means that the α_i 's cannot all be negative. This is the first restriction on the signs of the α_i 's.

It follows that $t^{(0)}(E_j) < t^{(0)}(E_k)$ for $j < k$ and therefore

$$0 \leq A \left(t^{(0)}(E_k) - t^{(0)}(E_j) \right) = \sum_{i=1}^m \gamma^{(i)-1} \alpha_i \left(t^{(i)}(E_k) - t^{(i)}(E_j) \right)$$

for $j < k$.

For a given j and k with $1 \leq j < k \leq n$, let

$$\begin{aligned} I(j, k) &= \{i \mid \sigma_i^{-1}(j) > \sigma_i^{-1}(k)\} \\ \bar{I}(j, k) &= \{i \mid \sigma_i^{-1}(j) < \sigma_i^{-1}(k)\} \end{aligned}$$

and consider the following sign pattern of the α_i 's:

$$\begin{aligned} \alpha_i < 0 & \quad \text{if and only if} \quad i \in I(j, k) \\ \alpha_i > 0 & \quad \text{if and only if} \quad i \in \bar{I}(j, k) \end{aligned}$$

Since $j < k$ if and only if $t^{(i)}(E_{\sigma_i(j)}) < t^{(i)}(E_{\sigma_i(k)})$, it also follows that $t^{(0)}(E_j) < t^{(0)}(E_k)$ if and only if $\sigma_i^{-1}(j) < \sigma_i^{-1}(k)$, and therefore $t^{(i)}(E_k) - t^{(i)}(E_j) > 0$ if and only if $i \in I(j, k)$. Consequently $\alpha_i(t^{(i)}(E_k) - t^{(i)}(E_j)) < 0$ for all $i \in I(j, k)$. Similarly, $t^{(0)}(E_j) > t^{(0)}(E_k)$ if and only if $\sigma_i^{-1}(j) > \sigma_i^{-1}(k)$, and therefore $t^{(i)}(E_k) - t^{(i)}(E_j) < 0$ if and only if $i \in \bar{I}(j, k)$. Consequently $\alpha_i(t^{(i)}(E_k) - t^{(i)}(E_j)) < 0$ for all $i \in \bar{I}(j, k)$. Therefore $\alpha_i(t^{(i)}(E_k) - t^{(i)}(E_j)) < 0$ for all i pairs, unconditionally.

As a result of this, the above sign pattern leads to the contradiction

$$0 \leq \sum_{i=1}^m \gamma^{(i)-1} \alpha_i \left(t^{(i)}(E_k) - t^{(i)}(E_j) \right) < 0$$

since the $\gamma^{(i)}$'s are all positive. The consequence is that each (j, k) pair with $j < k$ invalidates one possible sign pattern for the α_i 's.

The total number of sign patterns up for consideration is $2^{m-1} - 1$ since the all negative sign pattern has already been eliminated by the requirement $A > 0$. The number of (j, k) pairs is $\binom{n}{2}$, so if $\binom{n}{2} \geq 2^{m-1} - 1$, it is possible that all sign patterns are eliminated. The conclusion is that in such a case, the suggested sequence of observations of the events by the different observers is not possible.

It is interesting to consider the solutions of the Diophantine equation $\binom{n}{2} = 2^{m-1} - 1$. The known solutions are $\{(n = 3, m = 3), (n = 6, m = 5), (n = 91, m = 13)\}$. The first corresponds to restrictions on one space dimension, which has already been discussed. The second solution corresponds to restrictions on $3 + 1$ -dimensional space-time, while the third solution would correspond to restrictions on $11 + 1$ -dimensional space-time. Of course there could be other restrictions, since all that is necessary is that the inequality be satisfied.

Four Events in 3-Dimensional Space.

If there are four events in spacetime, labeled E_1, E_2, E_3, E_4 an observer will see the sequencing of these events in one of 24 possible ways. The goal of this section is to show that in 3-space, it is possible to choose these four events once and for all so they have a fixed sequence $t^{(0)}(E_1) < t^{(0)}(E_2) < t^{(0)}(E_3) < t^{(0)}(E_4)$ in reference frame 0, and that for any $\sigma \in \mathfrak{S}_4$, there is an inertial frame that sees these events in order σ , i.e. that

$$t^{(\sigma)}(E_{\sigma(1)}) < t^{(\sigma)}(E_{\sigma(2)}) < t^{(\sigma)}(E_{\sigma(3)}) < t^{(\sigma)}(E_{\sigma(4)}).$$

The velocity of this reference frame with respect to reference frame 1 is $\tilde{\mathbf{v}}^{(\sigma)}$ and

$$t^{(\sigma)}(E_i) = \gamma^{(\sigma)} \left(t^{(0)}(E_i) - \tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) \right)$$

for $i = 1, 2, 3, 4$. For this purpose, it is possible to take

$$\begin{aligned} \tilde{\mathbf{w}}^{(0)}(E_1) &= (0, 0, 0) & t^{(0)}(E_1) &= 1 \\ \tilde{\mathbf{w}}^{(0)}(E_2) &= (1, 0, 0) & t^{(0)}(E_2) &= 2 \\ \tilde{\mathbf{w}}^{(0)}(E_3) &= (0, 1, 0) & t^{(0)}(E_3) &= 3 \\ \tilde{\mathbf{w}}^{(0)}(E_4) &= (0, 0, 1) & t^{(0)}(E_4) &= 4 \end{aligned}$$

i.e. $t^{(0)}(E_i) = i$ for $i = 1, 2, 3, 4$. The Lorentz transformations then give

$$t^{(\sigma)}(E_i) = \gamma^{(\sigma)} \left(i - \tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) \right)$$

so

$$t^{(\sigma)}(E_{\sigma(i)}) = \gamma^{(\sigma)} \left(\sigma(i) - \tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{w}}^{(0)}(E_{\sigma(i)}) \right)$$

for $i = 1, 2, 3, 4$. Since $\gamma^{(\sigma)} > 0$, the conditions that $t^{(\sigma)}(E_{\sigma(i)}) < t^{(\sigma)}(E_{\sigma(i+1)})$ be met for $i = 1, 2, 3$ can be achieved by setting

$$t^{(\sigma)}(E_i) = \gamma^{(\sigma)} (s_\sigma + \sigma^{-1}(i))$$

for some constant s_σ to be determined that depends only on $\sigma \in \mathfrak{S}_4$, but not on i . Then

$$t^{(\sigma)}(E_{\sigma(i)}) = \gamma^{(\sigma)} (s_\sigma + i)$$

and the desired ordering follows since

$$\gamma^{(\sigma)} (s_\sigma + 1) < \gamma^{(\sigma)} (s_\sigma + 2) < \gamma^{(\sigma)} (s_\sigma + 3) < \gamma^{(\sigma)} (s_\sigma + 4).$$

It follows that the equations to be satisfied are

$$i - \tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) = s_\sigma + \sigma^{-1}(i)$$

for $i = 1, 2, 3, 4$. Since $\tilde{\mathbf{w}}^{(0)}(E_1) = 0$ it follows that

$$s_\sigma = 1 - \sigma^{-1}(1)$$

so that the remaining equations to be solved are

$$\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) = i - 1 + \sigma^{-1}(1) - \sigma^{-1}(i)$$

for $i = 2, 3, 4$, where the unknowns are the three components of $\tilde{\mathbf{v}}^{(\sigma)}$. However, this follows immediately from noting that $\tilde{\mathbf{w}}^{(0)}(E_2), \tilde{\mathbf{w}}^{(0)}(E_3), \tilde{\mathbf{w}}^{(0)}(E_4)$ are just the basis vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, respectively. This gives that the desired velocity for an observer to see the permutation σ on the temporal ordering of events E_1, E_2, E_3, E_4 is

$$\tilde{\mathbf{v}}^{(\sigma)} = (1 + \sigma^{-1}(1) - \sigma^{-1}(2), 2 + \sigma^{-1}(1) - \sigma^{-1}(3), 3 + \sigma^{-1}(1) - \sigma^{-1}(4)).$$

A slight problem here is that $|\tilde{\mathbf{v}}^{(\sigma)}|$ can be bigger than 1. In fact its maximum value is achieved when $\sigma^{-1}(1) = 4, \sigma^{-1}(2) = 3, \sigma^{-1}(3) = 2, \sigma^{-1}(4) = 1$, in which case $\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{v}}^{(\sigma)} = 56$. Replacing $\tilde{\mathbf{v}}^{(\sigma)}$ by $\tilde{\mathbf{v}}^{(\sigma)}/8$ ensures that $\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{v}}^{(\sigma)} < 1$, as desired. This is compensated for by replacing $\tilde{\mathbf{w}}^{(0)}(E_i)$ for $i = 1, 2, 3, 4$ by $8 \tilde{\mathbf{w}}^{(0)}(E_i)$.

The general case here is that in n -dimensional space there can be $n + 1$ events in space-time such that for every $\sigma \in \mathfrak{S}_{n+1}$ there is a reference frame such that the temporal ordering of the events is σ . Let \mathbf{e}_i denote the vector in \mathbb{R}^n with a 1 in coordinate i and 0's everywhere else. Now consider the events E_i where in reference frame 0, $\tilde{\mathbf{w}}^{(0)}(E_1) = \mathbf{0}$ and $\tilde{\mathbf{w}}^{(0)}(E_i) = \mathbf{e}_{i-1}$ for $i = 2, \dots, n + 1$ and $t_1(E_i) = i$ for $i = 1, \dots, n + 1$. Then choosing $\tilde{\mathbf{v}}^{(\sigma)}$ so that its i -th component is given by $\mathbf{v}_i^{(\sigma)} \cdot \mathbf{e}_i = i + \sigma^{-1}(1) - \sigma^{-1}(i + 1)$ means that an observer moving with velocity $\tilde{\mathbf{v}}^{(\sigma)}$ see the events E_i for $i = 1, \dots, n + 1$ occurring in time order σ . Here the maximum value of $\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{v}}^{(\sigma)}$ is achieved for $\sigma^{-1}(i) = n + 2 - i$ for $i = 1, \dots, n + 1$ in which case $\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{v}}^{(\sigma)} = i + (n + 1) - (n + 1 - i) = 2i$ so that

$$\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{v}}^{(\sigma)} = \sum_{i=1}^n (2i)^2 = \frac{4n(n-1)(2n-1)}{6} < 2n^3.$$

This allows replacing $\tilde{\mathbf{w}}^{(0)}(E_i)$ by $\tilde{\mathbf{w}}^{(0)}(E_i) \cdot 2n^3$ and $\tilde{\mathbf{v}}^{(\sigma)}$ by $\tilde{\mathbf{v}}^{(\sigma)}/(2n^3)$ for $i = 1, \dots, n + 1$ in order to assure that $\tilde{\mathbf{v}}^{(\sigma)} \cdot \tilde{\mathbf{v}}^{(\sigma)} < 1$.

Proposition. *In $n + 1$ -dimensional Minkowski space there exists an ordered set of $n + 1$ events such that for every $\sigma \in \mathfrak{S}_{n+1}$ there exists an inertial observation frame that sees these $n + 1$ events in the sequence σ .*

Four Observers in 3-Dimensional Space.

Consider now the question of whether an arbitrary number of events can be seen by a limited number of observers in any order. So suppose that there are n events in 3-dimensional space-time, E_i for $i = 1, \dots, n$. The goal of this section is to show that for any $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \mathfrak{S}_n$ there is some way to assign space and time coordinates to the E_i and to choose a set of reference frames R_0, R_1, R_2, R_3 so that reference frame R_i sees the events in time order σ_i for $i = 1, 2, 3, 4$.

Without loss of generality, it may be assumed that σ_0 is the identity permutation, $\sigma_0(i) = i$ for $i = 1, \dots, n$. In reference frame 0, assume that the time coordinate of the E_i is given by $t^{(0)}(E_i) = i$. Also assume that from the perspective of reference frame i , the velocity of R_i is given by $\tilde{\mathbf{v}}_i$, where

$$\begin{aligned}\tilde{\mathbf{v}}^{(0)} &= (0, 0, 0) \\ \tilde{\mathbf{v}}^{(1)} &= (v_1, 0, 0) \\ \tilde{\mathbf{v}}^{(2)} &= (0, v_2, 0) \\ \tilde{\mathbf{v}}^{(3)} &= (0, 0, v_3)\end{aligned}$$

for some $v \in \mathbb{R}$ with $0 < v_1, v_2, v_3 < 1$, i.e. $\tilde{\mathbf{v}}^{(0)} = \mathbf{0}$ and $\tilde{\mathbf{v}}^{(j)} = v_j \mathbf{e}_j$ for $j = 1, 2, 3$

In order that $t^{(j)}(E_i) = \gamma^{(j)} \sigma_j(i)$, which would mean that observer j sees the events in time order σ_j , it suffices that

$$\begin{aligned}\gamma^{(j)} \sigma_j(i) &= t_j(E_i) = \gamma^{(j)} \left(t_1(E_i) - \tilde{\mathbf{v}}^{(j)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) \right) \\ &= \gamma^{(j)} \left(i - v_{j-1} \mathbf{e}^{(j-1)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) \right)\end{aligned}$$

so that

$$\mathbf{e}^{(j-1)} \cdot \tilde{\mathbf{w}}^{(0)}(E_i) = v_{j-1}^{-1} (i - \sigma_j(i))$$

i.e.

$$\tilde{\mathbf{w}}^{(0)}(E_i) = ((i - \sigma_2(i))/v_1, (i - \sigma_3(i))/v_2, (i - \sigma_4(i))/v_3).$$

This then gives the space coordinates of the event E_i in the system of reference frame 1 that yield the desired space coordinates in the other reference frames.

Clearly, this generalizes to n -dimensional space, where any $n + 1$ observers can see any number of events in arbitrary orders.

Proposition. *For any $Q \subset \mathfrak{S}_k$ with $|Q| = n + 1$, there exists a set of k events in $n + 1$ -dimensional Minkowski space and a set of $n + 1$ inertial observation frames that realize the set Q , i.e. for each $\sigma \in Q$ there is a frame that sees the k events in order σ .*

Five Observers of Six Events in 3-Dimensional Space: A Computer Search.

In this case, it is at least possible to eliminate some selections of five permutations from \mathfrak{S}_6 simply on the basis of sign patterns.

The following tables contain the results of a computer search for disallowed sets of observation sequences for five inertial observers, each observing the same set of six events, in potentially different orders. Without any loss of generality, it may be assumed that one of the observers sees the events in serial order $(1, 2, 3, 4, 5, 6)$ so that rather than $\binom{720}{5} = 1,590,145,128,144$ sets of 5 permutations to search through, it is only necessary to search through a set of $\binom{719}{4} = 11,042,674,501$, which is a much more tractable number. As in the case of 5 events with 5 observers, any one of the observers can be designated as having the identity permutation, so many of these sets are equivalent. The number of distinct equivalence classes of sets is 2,208,534,929 by direct count. Note that $5 \cdot 2208534929 - 11042674501 = 144$, which is total number of 5-cycles in \mathfrak{S}_6 , i.e. the number of distinct subgroups of \mathfrak{S}_6 of order 5. (To see this note that there are 6 ways of picking out 5 elements of $\{1, 2, 3, 4, 5, 6\}$ and that each such set of 5 elements gives rise to $4! = 24$ distinct 5-cycles.)

The computer search for disallowed sets was based on eliminating all possible sign patterns for the relative velocities of the different observers. The total number of disallowed sign patterns found by the computer search was 294. A closer examination of these 294 answers shows that the actual number can be substantially reduced. If $S = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ is any disallowed pattern, then so is $S\sigma = \{\pi_0\sigma, \pi_1\sigma, \pi_2\sigma, \pi_3\sigma, \pi_4\sigma\}$ for any permutation $\sigma \in \mathfrak{S}_6$. In particular if $\pi_0 = (1, 2, 3, 4, 5, 6)$ is the identity permutation, the taking $\sigma = \pi_i^{-1}$ for $i = 1, 2, 3, 4$ will give other disallowed sets permutations where one element of the set is the identity that are essentially the same as S . Generally, this procedure gives 5 disallowed sets from each one, which will result in a reduction in the number by a factor of 5. However, there is the possibility that the set S will actually be a group of permutations of order 5, in which case $S\pi_i^{-1} = S$ for each $\pi_i \in S$. In the present case, there are four such groups of order 5, leaving a total of 290 sets which are not groups, and thus there are 58 essential remaining cases.

Just as in the case of five observers of five events, in the case of five observers of six events, where 294 of the 11,042,674,501 were found to be disallowed, it is not known whether all of the remaining 11,042,674,207 cases can all occur. All that is known is that there are allowable sign patterns of the α_j 's (which are the coefficients in the linear relation among the velocities), that might make the pattern of permutations possible.

Given the complexity associated with constructing point sets of size five that realize sets of five permutations from \mathfrak{S}_5 , it seems a bit daunting to try larger constructions. However, in principle it is clear how to generalize the prior construction. Starting with four observers, one in the rest frame and the other three moving along orthogonal axes, a set of six points in $3 + 1$ -dimensional spacetime can be constructed that realize four of the four permutations. The final observer frame can then be constructed if a set of linear inequalities can be satisfied. The method used above now leads to two "gap equations", instead of one, namely $g_4 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \beta$ and $g_5 = \alpha'_1 g_1 + \alpha'_2 g_2 + \alpha'_3 g_3 + \beta'$ that need to be solved with $g_i > 0$ for $i = 1, 2, 3, 4, 5$.

Another phenomenon is that time reversal of a disallowed set should yield another disallowed set. The time reversal of a set S is $\pi_r S \pi_r$ where $\pi_r = (6, 5, 4, 3, 2, 1)$. The right multiplication by π_r accomplishes the time reversal itself, by reversing the order of the observations for each observer. The left multiplication by π_r simply

relabels the observations. Of the 58 cases to consider, 8 were time reversal invariant, while 50 were not.

The 8 time reversal invariant cases are listed below. It is interesting to note that one of these cases is also invariant under taking the inverse of each permutation in the set.

- 1: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (2, 6, 1, 5, 4, 3), (4, 3, 2, 6, 1, 5), (5, 4, 2, 1, 3, 6)\}$
- 2: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (3, 4, 5, 6, 1, 2), (5, 4, 2, 1, 3, 6), (5, 6, 1, 2, 3, 4)\}$
- 3: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 4, 2, 3), (2, 5, 4, 6, 1, 3), (4, 5, 3, 1, 2, 6), (4, 6, 1, 3, 2, 5)\}$
- 4: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 4, 2, 3), (3, 6, 1, 4, 5, 2), (4, 5, 3, 1, 2, 6), (5, 2, 3, 6, 1, 4)\}$ self inverse
- 5: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 4, 1, 2, 3, 5)\}$
- 6: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (4, 3, 5, 1, 6, 2), (5, 1, 6, 2, 4, 3), (6, 4, 1, 2, 3, 5)\}$
- 7: $\{(1, 2, 3, 4, 5, 6), (3, 6, 1, 4, 5, 2), (4, 2, 3, 6, 5, 1), (5, 2, 3, 6, 1, 4), (6, 2, 1, 4, 5, 3)\}$
- 8: $\{(1, 2, 3, 4, 5, 6), (4, 1, 6, 3, 5, 2), (4, 2, 3, 6, 5, 1), (5, 2, 4, 1, 6, 3), (6, 2, 1, 4, 5, 3)\}$

None of the four groups of order five were time reversal invariant. Thus, excluding the groups, there are 25 essentially different cases that are not time reversal invariant. There are essentially now only two different groups of order five. None of these cases are invariant under inversion. The complete results are given in appendix 1.

Five Observers of Five Events in 3-Dimensional Space.

Let $Q = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\} \subset \mathfrak{S}_5$ be a set of distinct event orderings of a set $\{E_1, E_2, E_3, E_4, E_5\}$ of five (spacelike separated) events for 5 observers in 5 different inertial frames. Without loss of generality, assume that π_0 is the identity permutation in the rest frame, i.e. $\pi_0 = (1, 2, 3, 4, 5)$. In this rest frame, event E_i occurs at time t_i and at spatial coordinate \tilde{w}_i for $i \in \{1, 2, 3, 4, 5\}$. Since the ordering of events in the rest frame is the identity permutation, this is equivalent to asserting that $t_1 < t_2 < t_3 < t_4 < t_5$. If observer j is moving at velocity $\tilde{v}^{(j)}$ from the perspective of the rest frame (so $\tilde{v}^{(0)} = \mathbf{0}$), then the time of event E_i in frame j is

$$T_i^{(j)} = \gamma^{(j)} (t_i - \tilde{v}^{(j)} \cdot \tilde{w}_i)$$

where $\gamma^{(j)} = (1 - \tilde{v} \cdot \tilde{v})^{-1/2} > 0$.

In what follows it will be useful to deal with the “relativized time” of event i for observer j defined as

$$t_i^{(j)} = t_i - \tilde{v}^{(j)} \cdot \tilde{w}_i$$

which is the same as $T_i^{(j)}$ except for the Lorentz dilation factor $\gamma^{(j)}$. For the purposes of analyzing relative orders of events being observed in frame j , the constant factor $\gamma^{(j)}$ is irrelevant, and only makes the formulas more complicated. Also, in what follows, it will be mathematically possible to have frame velocities larger than 1, which of course is physically impossible. It is always possible to divide all the velocities by a large constant N and then multiply the spatial coordinates of all events by the same constant. This way, in the end, all the velocities can be made less than 1 with the observed (relativized) event times being kept the same, and hence their ordering in frame j also is unchanged.

In addition to assuming that $\tilde{v}^{(0)} = \mathbf{0}$, if three of the remaining four velocity vectors span all of \mathbb{R}^3 (i.e. they don’t all come from a some smaller dimensional

subspace) it is also possible to change these three non-zero velocities to be any set of three linearly independent velocities by making corresponding adjustments to the coordinates of the events in spacetime. Assuming $\tilde{v}^{(1)}$, $\tilde{v}^{(2)}$, and $\tilde{v}^{(3)}$ are linearly independent, if $\{\tilde{V}^{(1)}, \tilde{V}^{(2)}, \tilde{V}^{(3)}\}$ is any other set of linearly independent velocity vectors, then there is an invertible matrix R such that $\tilde{V}^{(j)} = \tilde{V}^{(j)} R$ for $j = 1, 2, 3$. Now taking $W_i = R^{-1} w_i$ for all i and $\tilde{V}^{(j)} = \tilde{V}^{(j)} R$ for all j gives $t_i - \tilde{V}^{(j)} \cdot \tilde{W}_i = t_i - \tilde{v}^{(j)} \cdot \tilde{w}_i = t_i^{(j)}$ for all i and j . Thus without any loss of generality, it can be assumed that $\tilde{v}^{(2)}$, and $\tilde{v}^{(3)}$ are orthogonal unit vectors in \mathbb{R}^3 .

Reducing the search to satisfying a single inequality.

What is desired is that

$$t_{\pi_j(i+1)}^{(j)} > t_{\pi_j(i)}^{(j)}$$

for $i = 1, 2, 3, 4$. Now for $j = 0, 1, 2, 3$ let

$$F_j : \{1, 2, 3, 4, 5\} \rightarrow \mathbb{R}$$

be a set of increasing functions. It may be useful to write

$$h_i^{(j)} = F_j(i+1) - F_j(i)$$

for $j = 0, 1, 2, 3$ and $i = 1, 2, 3, 4$ with the condition that the $h_i^{(j)}$'s are all positive, so that $F_j(i) = F_j(1) + \sum_{l=1}^{i-1} h_l^{(j)}$ for $2 \leq i \leq 5$. This in turn implies that

$$F_j(i_1) - F_j(i_0) = \sum_{l=i_0}^{i_1-1} h_l^{(j)}$$

for $1 \leq i_0 < i_1 \leq 5$.

Suppose that the spacetime coordinates of event E_i in the rest frame are

$$t_i = F_0(i)$$

and

$$\tilde{w}_i = (w_{i,1}, w_{i,2}, w_{i,3}) = (F_0(i) - F_1(\pi_1^{-1}(i)), F_0(i) - F_2(\pi_2^{-1}(i)), F_0(i) - F_3(\pi_3^{-1}(i))).$$

Then clearly $t_i^{(0)} = F_0(i)$ so $F_0(i+1) = t_{\pi_0(i+1)}^{(0)} > t_{\pi_0(i)}^{(0)} = F_0(i)$ for $i = 1, 2, 3, 4$ since F_0 is an increasing function. Now set

$$\tilde{v}^{(1)} = (1, 0, 0)$$

$$\tilde{v}^{(2)} = (0, 1, 0)$$

$$\tilde{v}^{(3)} = (0, 0, 1)$$

so that

$$t_i^{(1)} = t_i - \tilde{v}^{(1)} \cdot \tilde{w}_i = F_1(\pi_1^{-1}(i))$$

for $i = 1, 2, 3, 4, 5$, and therefore $t_{\pi_1(i)}^{(1)} = F_1(i)$. Since F_1 is an increasing function, it now follows that $t_{\pi_1(i+1)}^{(1)} > t_{\pi_1(i)}^{(1)}$ for $i = 1, 2, 3, 4$. Similarly $t_{\pi_2(i)}^{(2)} = F_2(i)$ and

$t_{\pi_3(i)}^{(3)} = F_3(i)$ and therefore $t_{\pi_2(i+1)}^{(2)} > t_{\pi_2(i)}^{(2)}$ and $t_{\pi_3(i+1)}^{(3)} > t_{\pi_3(i)}^{(3)}$ for $i = 1, 2, 3, 4$ since F_2 and F_3 are also increasing functions. Thus, in general

$$t_{\pi_j(i+1)}^{(j)} - t_{\pi_j(i)}^{(j)} = F_j(i+1) - F_j(i) = h_i^{(j)} > 0$$

for $i = 1, 2, 3, 4$ and $j = 0, 1, 2, 3$, by design

What remains is to consider the event ordering for the fifth observer, whose velocity vector with respect to the rest frame is $\tilde{v}^{(4)} = \tilde{u} = (u_1, u_2, u_3)$ with components that are to be determined. To this end

$$\begin{aligned} t_i^{(4)} &= t_i - \tilde{v}^{(4)} \cdot \tilde{w}_i \\ &= F_0(i) - u_1 (F_0(i) - F_1(\pi_1^{-1}(i))) - u_2 (F_0(i) - F_2(\pi_2^{-1}(i))) \\ &\quad - u_3 (F_0(i) - F_3(\pi_3^{-1}(i))) \\ &= F_0(i) (1 - u_1 - u_2 - u_3) + u_1 F_1(\pi_1^{-1}(i)) + u_2 F_2(\pi_2^{-1}(i)) + u_3 F_3(\pi_3^{-1}(i)) \end{aligned}$$

and therefore

$$\begin{aligned} t_{\pi_4(i)}^{(4)} &= F_0(\pi_4(i)) (1 - u_1 - u_2 - u_3) + u_1 F_1(\pi_1^{-1}(\pi_4(i))) \\ &\quad + u_2 F_2(\pi_2^{-1}(\pi_4(i))) + u_3 F_3(\pi_3^{-1}(\pi_4(i))) \end{aligned}$$

for $i = 1, 2, 3, 4, 5$, and note that the $t^{(4)}$'s are all linear functions of the u 's. Now consider the gap between the values of the $t^{(4)}$'s, by setting

$$\begin{aligned} g_i &= t_{\pi_4(i+1)}^{(4)} - t_{\pi_4(i)}^{(4)} \\ &= (F_0(\pi_4(i+1)) - F_0(\pi_4(i))) (1 - u_1 - u_2 - u_3) \\ &\quad + u_1 (F_1(\pi_1^{-1}(\pi_4(i+1))) - F_1(\pi_1^{-1}(\pi_4(i)))) \\ &\quad + u_2 (F_2(\pi_2^{-1}(\pi_4(i+1))) - F_2(\pi_2^{-1}(\pi_4(i)))) \\ &\quad + u_3 (F_3(\pi_3^{-1}(\pi_4(i+1))) - F_3(\pi_3^{-1}(\pi_4(i)))) \end{aligned}$$

for $i = 1, 2, 3, 4$. The conditions $t_{\pi_4(i+1)}^{(4)} > t_{\pi_4(i)}^{(4)}$ for $i = 1, 2, 3, 4$ will be satisfied if and only if all the g_i 's are positive. However, the g_i 's are linear functions of u_1 , u_2 , and u_3 , so there is a linear relation between the g_i 's. Assuming nonsingularity of the linear system (which will be generically true, and can be guaranteed by slight variations in the parameters), there is a linear relationship

$$g_4 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \beta$$

for some constants, α_1 , α_2 , α_3 , and β that depend only on F_0 , F_1 , F_2 , and F_3 and on π_1 , π_2 , π_3 , and π_4 . If either β is positive or the α 's are not all negative, then it is possible to take all the g_i 's to be positive. Since the g 's are all linear functions of the u 's, it will then be possible to find the desired velocity vector $v^{(4)}$ so that the fifth observer will see the events in order π_4 .

Write

$$g_i = b_i + \tilde{a}_i \cdot \tilde{u}$$

where $\tilde{v}^{(4)} = \tilde{u} = (u_1, u_2, u_3)$ and $\tilde{a}_i = (a_{i,1}, a_{i,2}, a_{i,3})$ with

$$\begin{aligned} a_{i,j} &= (F_j(\pi_j^{-1}(\pi_4(i+1))) - F_j(\pi_j^{-1}(\pi_4(i)))) + (F_0(\pi_4(i)) - F_0(\pi_4(i+1))) \\ &= w_{\pi_4(i+1),j} - w_{\pi_4(i),j} \end{aligned}$$

for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ and

$$\begin{aligned} b_i &= F_0(\pi_4(i+1)) - F_0(\pi_4(i)) \\ &= t_{\pi_4(i+1)} - t_{\pi_4(i)} \end{aligned}$$

for $i = 1, 2, 3, 4$. Let A be the 4×3 matrix with entries $(a_{i,j})$ as above, and set $\tilde{g} = (g_1, g_2, g_3, g_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ as row vectors. Then the above equations can be written as

$$\tilde{g}^\tau = A \tilde{u}^\tau + \tilde{b}^\tau.$$

Since A has more rows than columns, it should generally be possible to write the last row of A as a linear combination of the top three rows. Write

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \quad \text{and} \quad A_0 = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix}$$

and ask for a solution $\tilde{\alpha} = (\alpha_1 \ \alpha_2 \ \alpha_3)$ to

$$A_0 \tilde{\alpha}^\tau = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a_{4,1} \\ a_{4,2} \\ a_{4,3} \end{pmatrix}$$

so that $\tilde{\alpha}' A = 0$ where $\tilde{\alpha}' = (\alpha_1 \ \alpha_2 \ \alpha_3 \ -1)$. Then

$$\alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 - g_4 = \tilde{\alpha} \tilde{g}^\tau = \tilde{\alpha}' A \tilde{u}^\tau + \tilde{\alpha}' \tilde{b}^\tau = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 - b_4$$

i.e.

$$g_4 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + (b_4 - \alpha_1 b_1 - \alpha_2 b_2 - \alpha_3 b_3)$$

as the desired relation between the g_i 's. In particular $\beta = b_4 - \alpha_1 b_1 - \alpha_2 b_2 - \alpha_3 b_3$. Note that this depends on the nonsingularity of the system of equations, but in general, by varying the F_j 's, this can be accomplished.

Of course, if the system of equations is singular, or if $\alpha_1, \alpha_2, \alpha_3$, and β are all negative, this program will fail. In that case, a few options are available. Note that this whole approach started with picking out three of the four elements of $\{\pi_1, \pi_2, \pi_3, \pi_4\}$ and setting the x, y , and z coordinates (in the rest frame) of the events $\{E_1, E_2, E_3, E_4, E_5\}$ to make those three permutations realizable. Picking a different set of three permutations out of the four will lead to a different set of equations to solve for the fourth velocity vector, and may lead to a solution for the fourth permutation.

Another variation is to change the labelling on the events and pick a different observer to be designated as the rest frame. This means that the set of permutations $Q = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ is equivalent to the set $Q\sigma = \{\pi_0\sigma, \pi_1\sigma, \pi_2\sigma, \pi_3\sigma, \pi_4\sigma\}$ for

any σ . In particular, taking $\sigma = \pi_i^{-1}$ for $i = 1, 2, 3, 4$ will give equivalent sets to Q where one of the permutations is π_0 , and this will allow the entire theory above to be applied.

Another possible variation to note is that if a set Q of permutations is unachievable, then so is its time reversal $\pi^{(0)}Q\pi^{(0)}$, where $\pi^{(0)} = (5, 4, 3, 2, 1)$. This may also lead to additional disqualified sets.

Finally, there is always the option of varying the F_j 's as long as they are all kept as increasing functions. This is equivalent to not requiring the intervals between successive events in frames 1, 2, and 3 to be equally spaced. Note that the $a_{i,j}$'s themselves are simply linear functions of the $h_i^{(j)}$'s and that the $h_i^{(j)}$'s are all required to be positive. In particular the initial values $F_j(1)$ don't really matter since the F_j 's only appear as differences.

Results of a computer search.

The ideas above were programmed and tested to see if every set of five distinct permutations from S_5 was realizable. This is a non-trivial, but also quite doable, search problem. This sort of search problem is well within the range of computer technology, the search space being of size 7,940,751. In principle this can be cut down to a search of size 1,588,155, although that was not actually done.

For each of the 7,940,751 choices of five distinct elements from \mathfrak{S}_5 and for each quadruple (F_0, F_1, F_2, F_3) of increasing functions from $\{1, 2, 3, 4, 5\}$ to \mathbb{R} , the above procedural description yields up to twenty different linear relations of the form $g_4 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \beta$, and any one of them that allows a solution with all the g_i 's positive will yield a set of five points in Minkowski space and a set of five frame velocities that will realize the desired set of permutations. The twenty possibilities come from five equivalent sets to the set of permutations times four choices of three out of four permutations to choose for orthogonal velocity vectors.

Note that one of the things that can go wrong here is that the set of equations for the α_i 's, which is of the form $\tilde{a}_4 = \alpha_1 \tilde{a}_1 + \alpha_2 \tilde{a}_2 + \alpha_3 \tilde{a}_3$, may in fact be singular, and therefore not solvable, which is why there may actually be less than twenty different linear relations to actually test.

The simplest choices of increasing functions are simply linear, i.e

$$F_j(x) = m_j x$$

where $m_j > 0$ can be any positive real number. There seems to be no systematic way of choosing a good set of m_i 's, but if one set of m_i 's fails to yield any solvable linear relations, another set of m_i 's can be readily tried, and that new set may well succeed. One somewhat bad choice, however, is taking $m_1 = m_2 = m_3 = 1$ since that doesn't really lead to twenty distinct linear relations due to the symmetry of the problem, and in fact this leads only to 5 different linear relations.

A run starting with $(m_0, m_1, m_2, m_3) = (1, 300, 200, 300)$ yielded a collection of only 333 permutation sets that were not realized.¹ Some experimentation with different F_j 's now helps a lot. The following tabulated function

¹Some further analysis of the answer yielded an interesting fact. For 160 of these cases, the linear system of equations for the α_i 's was singular regardless of how the permutations were reordered or whether a equivalent set was chosen. Also, these 160 cases were the same, regardless of how the m_i 's were changed, which clearly suggest that these 160 cases are intrinsically singular and that there is an underlying identity. For the remaining 173 cases, the system of equations for the α_i 's were solvable, and they were solvable regardless of how the permutations were ordered

$F_0(1) = 1.0$	$F_0(2) = 2.0$	$F_0(3) = 3.0$	$F_0(4) = 4.0$	$F_0(5) = 5.0$
$F_1(1) = 1.0$	$F_1(2) = 2.0$	$F_1(3) = 3.0$	$F_1(4) = 7.0$	$F_1(5) = 8.0$
$F_2(1) = 1.0$	$F_2(2) = 2.0$	$F_2(3) = 24.0$	$F_2(4) = 25.0$	$F_2(5) = 26.0$
$F_3(1) = 1.0$	$F_3(2) = 64.0$	$F_3(3) = 65.0$	$F_3(4) = 66.0$	$F_3(5) = 67.0$

realizes all but 3 cases of the 333 remaining permutation sets. A closer examination shows that each of these last three cases are cyclic subgroups of \mathfrak{S}_5 of order 5, and that two of these groups of order 5 are time reversal conjugates of each other. Some further experimentation leads to the following tabulated function

$F_0(1) = 1.0$	$F_0(2) = 2.0$	$F_0(3) = 3.0$	$F_0(4) = 4.0$	$F_0(5) = 5.0$
$F_1(1) = 1.0$	$F_1(2) = 23.0$	$F_1(3) = 24.0$	$F_1(4) = 25.0$	$F_1(5) = 26.0$
$F_2(1) = 4.0$	$F_2(2) = 5.0$	$F_2(3) = 6.0$	$F_2(4) = 7.0$	$F_2(5) = 8.0$
$F_3(1) = 1.0$	$F_3(2) = 2.0$	$F_3(3) = 65.0$	$F_3(4) = 66.0$	$F_3(5) = 67.0$

which realizes the two remaining permutation sets that are time reversals of each other.

This leaves one lone case still unresolved. This remaining case is the set

$$Q_0 = \{(1, 2, 3, 4, 5), (2, 3, 4, 5, 1), (3, 4, 5, 1, 2), (4, 5, 1, 2, 3), (5, 1, 2, 3, 4)\}$$

which is its own time reverse, and is actually invariant under inversion too, since in fact this set of permutation is a cyclic group of order 5.

So out of an initial set of 7,940,751 choices of five distinct elements from \mathfrak{S}_5 (containing the identity), all but at most one set of permutations can be realized, i.e. for any such set of five elements of \mathfrak{S}_5 , there are five events in three dimensional space time and five observer frames such that each observer sees that the five events in the specified permutation.

Analysis of the final case.

It is useful to let $\pi = (2, 3, 4, 5, 1)$ be a generator of the cyclic group under consideration. Then the set of permutations to be considered is just $Q_0 = \{\pi^0, \pi^1, \pi^2, \pi^3, \pi^4\}$, where $\pi_j = \pi^j$. If a set of representatives of $\mathbb{Z}/5\mathbb{Z}$ is taken to be $\{1, 2, 3, 4, 5\}$ (rather than the more usual $\{0, 1, 2, 3, 4\}$), then $\pi_j(i) = i + j \bmod 5$. In particular $\pi_j^{-1}(\pi_4(i)) = i + 4 - j \bmod 5$ and $\pi_j^{-1}(\pi_4(i + 1)) = i - j \bmod 5$, so

$$\begin{aligned} F_j(\pi_j^{-1}(\pi_4(i))) - F_j(\pi_j^{-1}(\pi_4(i + 1))) &= F_j(i + 4 - j \bmod 5) - F_j(i - j \bmod 5) \\ &= \begin{cases} h_1^{(j)} + h_2^{(j)} + h_3^{(j)} + h_4^{(j)} & \text{if } i - j \equiv 1 \bmod 5; \\ -h_{(i-j \bmod 5)-1}^{(j)} & \text{if } 2 \leq (i - j \bmod 5) \leq 5. \end{cases} \end{aligned}$$

Similarly, $\pi_4(i) = i + 4 \bmod 5$ and $\pi_4(i + 1) = i \bmod 5$, so

$$\begin{aligned} F_0(\pi_4(i)) - F_0(\pi_4(i + 1)) &= F_0(i + 4 \bmod 5) - F_0(i \bmod 5) \\ &= \begin{cases} h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} & \text{if } i = 1; \\ -h_{i-1}^{(0)} & \text{if } 2 \leq i \leq 5. \end{cases} \end{aligned}$$

or whether a equivalent set was used. They just led to a set of equations of the type $g_4 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \beta$ where $\alpha_i < 0$ for $i = 1, 2, 3$ and $\beta < 0$, and the signs of the α 's and β didn't really depend on the choice of the m_i 's. However, of the 160 singular cases, only 60 appeared with their time reverse, meaning that for 100 of the cases, where the system was found to be singular, the time reverse was found to be unachievable (and in particular also gave rise to a nonsingular system), which means that this set of permutations was also unachievable.

Therefore

$$a_{i,j} = (F_j(\pi_j^{-1}(\pi_4(i+1))) - F_j(\pi_j^{-1}(\pi_4(i)))) + (F_0(\pi_4(i)) - F_0(\pi_4(i+1)))$$

$$= \begin{cases} -h_1^{(j)} - h_2^{(j)} - h_3^{(j)} - h_4^{(j)} - h_{i-1}^{(0)} & \text{if } i-j \equiv 1 \pmod{5} \text{ and } 2 \leq i \leq 5; \\ h_4^{(j)} + h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} & \text{if } 2 \leq (i-j \pmod{5}) \leq 5 \text{ and } i=1; \\ h_{(i-j \pmod{5})-1}^{(j)} - h_4^{(0)} & \text{if } 2 \leq (i-j \pmod{5}) \leq 5 \text{ and } 2 \leq i \leq 5; \end{cases}$$

where the case $i-j \equiv 1 \pmod{5}$ and $i=1$ has been omitted since this implies $j=0$, which is not relevant for what follows. These are the entries of the matrix 4×3 A which has entries $(a_{i,j})$ for $i=1,2,3,4$ and $j=1,2,3$, so that

$$A = \begin{pmatrix} h_4^{(1)} + h_1^{(0)} + h_2^{(0)} & h_3^{(2)} + h_1^{(0)} + h_2^{(0)} & h_2^{(3)} + h_1^{(0)} + h_2^{(0)} \\ +h_3^{(0)} + h_4^{(0)} & +h_3^{(0)} + h_4^{(0)} & +h_3^{(0)} + h_4^{(0)} \\ -h_1^{(1)} - h_2^{(1)} - h_3^{(1)} & h_4^{(2)} - h_1^{(0)} & h_3^{(3)} - h_1^{(0)} \\ -h_4^{(1)} - h_1^{(0)} & h_4^{(2)} - h_1^{(0)} & h_3^{(3)} - h_1^{(0)} \\ h_1^{(1)} - h_2^{(0)} & -h_1^{(2)} - h_2^{(2)} - h_3^{(2)} & h_4^{(3)} - h_2^{(0)} \\ -h_4^{(2)} - h_2^{(0)} & -h_4^{(2)} - h_2^{(0)} & h_4^{(3)} - h_2^{(0)} \\ h_2^{(1)} - h_3^{(0)} & h_1^{(2)} - h_3^{(0)} & -h_1^{(3)} - h_2^{(3)} - h_3^{(3)} \\ & & -h_4^{(3)} - h_3^{(0)} \end{pmatrix}$$

and

$$\begin{aligned} b_1 &= -h_1^{(0)} - h_2^{(0)} - h_3^{(0)} - h_4^{(0)} \\ b_2 &= h_1^{(0)} \\ b_3 &= h_2^{(0)} \\ b_4 &= h_3^{(0)} \end{aligned}$$

since $b_i = F_0(\pi_4(i+1)) - F_0(\pi_4(i))$ for $i=1,2,3,4$.

In general for a 4×3 matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}$$

there are four 3×3 matrices obtained by deleting a single row

$$\begin{aligned} A_1 &= \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} A_2 = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ A_3 &= \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} A_4 = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{aligned}$$

with the following general linear relation between the rows of A

$$\begin{aligned} 0 &= \det(A_1) \cdot (a_{1,1}, a_{1,2}, a_{1,3}) - \det(A_2) \cdot (a_{2,1}, a_{2,2}, a_{2,3}) \\ &\quad + \det(A_3) \cdot (a_{3,1}, a_{3,2}, a_{3,3}) - \det(A_4) \cdot (a_{4,1}, a_{4,2}, a_{4,3}). \end{aligned}$$

Writing $D_i = \det(A_i)$ for $i = 1, 2, 3, 4$, and setting

$$\begin{aligned}\tilde{a}_1 &= (a_{1,1}, a_{1,2}, a_{1,3}) \\ \tilde{a}_2 &= (a_{2,1}, a_{2,2}, a_{2,3}) \\ \tilde{a}_3 &= (a_{3,1}, a_{3,2}, a_{3,3}) \\ \tilde{a}_4 &= (a_{4,1}, a_{4,2}, a_{4,3})\end{aligned}$$

so that if $D_4 \neq 0$ then

$$\tilde{a}_4 = \alpha_1 \tilde{a}_1 + \alpha_2 \tilde{a}_2 + \alpha_3 \tilde{a}_3$$

with

$$\alpha_1 = D_1/D_4 \quad \alpha_2 = -D_2/D_4 \quad \alpha_3 = D_3/D_4.$$

It is a straightforward (but uncomfortably large) algebraic calculation to find the D_k 's as functions of the $h_i^{(j)}$'s, and the details of this have been relegated to appendix 2. However, the point of the whole computation is to show that $D_1, D_3 < 0$ and $D_2, D_4 > 0$. Therefore $\alpha_1, \alpha_2, \alpha_3 < 0$. Also

$$\beta = b_4 - \alpha_1 b_1 - \alpha_2 b_2 - \alpha_3 b_3$$

and the computation of β (also relegated to appendix 2), shows that $\beta < 0$, as well.

Since $\alpha_i < 0$ for $i = 1, 2, 3$ and $\beta < 0$, it follows that $g_4 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \beta < 0$ if $g_i > 0$ for $i = 1, 2, 3$, and therefore there is no velocity vector $\tilde{v}^{(4)} = \tilde{u} = (u_1, u_2, u_3)$ that realizes the fourth permutation.

Theorem. *Let $Q \subset \mathfrak{S}_5$ with $|Q| = 5$ and $\pi_0 \in Q$. If $Q \neq Q_0$, then there exist a set of 5 points in spacetime and a set of 5 inertial reference frames that realize the set Q . If $Q = Q_0$, then such a set of 5 points in spacetime and a set of 5 reference frames does not exist.*

Implications for five reference frames and six points.

With a single forbidden configuration of five permutation on five elements, it is easy to construct sets of five forbidden permutations on six elements by simply slipping in the sixth element anywhere in each of the five permutations comprising Q_0 .

In principle this new event is viewed as an insertion of a new number in each extant permutation giving now a set of five elements of \mathfrak{S}_6 . This insertion can occur in any one of six possible points for each π^i , and all possibilities must be considered. There are 6^5 such possibilities, however for each possibility, it is necessary to convert it to "standard" form where one of the permutations is the identity. There five ways to do this corresponding to five different choices of which observer sees the identity permutation, making a total of $5 \cdot 6^5 = 38,880$ possible cases to consider. Note that when this is done, it may be that some of cases are the same. The way that duplicates are eliminated is by sorting the set of cases.

When this is done, and all the duplication is eliminated, there are 7676 subsets of \mathfrak{S}_6 of cardinality 5 that contain the identity permutation remaining. All of these represent forbidden configurations in addition to the 294 forbidden configurations already found by considering sign patterns. It is interesting that this new collection of 7676 forbidden sets is completely disjoint from the prior collection of 294 forbidden sets.

As discussed above, if $S = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ is any disallowed pattern, then so is $S\sigma$ and taking $\sigma = \pi_i^{-1}$ for $i = 1, 2, 3, 4$ will give other disallowed sets permutations where one element of the set is the identity. These the same as S , and generally, this procedure gives 5 disallowed sets from each one, however there is the possibility that the set S will actually be a group of permutations of order 5 contained in S_6 . In the present case, of the 7676 forbidden cases coming from Q_0 , there are six such groups of order 5, leaving a total of 7670 sets which are not groups, and a total of 1534 essential remaining cases. Of the 1540 essentially different cases, only six were invariant under time reversal (and interestingly one of these is actually invariant under permutation inversion), while 1534 were not. The total number of cases to consider then is $6 + 1534/2 = 773$. It would be too long, and probably not of great interest to list all 773 cases, since they are all easily constructed from the basic Q_0 set of five elements of \mathfrak{S}_5 . It may be of some interest to list the six time reversal invariant sets. They are:

- 1: $\{(1, 2, 3, 4, 5, 6), (3, 4, 5, 1, 6, 2), (4, 5, 1, 6, 2, 3), (5, 1, 6, 2, 3, 4), (6, 2, 3, 4, 5, 1)\}$
- 2: $\{(1, 2, 3, 4, 5, 6), (3, 4, 5, 1, 6, 2), (4, 5, 6, 1, 2, 3), (5, 1, 6, 2, 3, 4), (6, 2, 3, 4, 5, 1)\}$
- 3: $\{(1, 2, 3, 4, 5, 6), (2, 3, 4, 5, 6, 1), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 1, 2, 3, 4, 5)\}$ self inverse
- 4: $\{(1, 2, 3, 4, 5, 6), (2, 4, 3, 5, 6, 1), (4, 3, 5, 6, 1, 2), (5, 6, 1, 2, 4, 3), (6, 1, 2, 4, 3, 5)\}$
- 5: $\{(1, 2, 3, 4, 5, 6), (2, 4, 3, 5, 6, 1), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 1, 2, 4, 3, 5)\}$
- 6: $\{(1, 2, 3, 4, 5, 6), (2, 3, 4, 5, 6, 1), (4, 3, 5, 6, 1, 2), (5, 6, 1, 2, 4, 3), (6, 1, 2, 3, 4, 5)\}$

A seemingly much more complicated question is whether there are other unrealizable subsets of \mathfrak{S}_6 of size five that are unrealizable that are not included in the $7676 + 294 = 7970$ cases already considered.

CONCLUSION

This note represents an initial attempt to capture the combinatorial nature of special relativity. It is an interesting question whether starting with the combinatorial restrictions imposed by special relativity eventually lead to Minkowsky space-time.

In 3+1-dimensional spacetime with 5 observers of 5 spacelike separated events, it is striking that of the 7,940,751 possible cases to consider, all but exactly one are realizable. The proof is based on a computer construction of the 7,940,750 realizable cases along with a computer algebra computation to show that the remaining case is impossible. One might hope that a more conceptual proof could be found. With 5 observers of 6 spacelike separated events, along with extending the impossible case of 5 events to an arbitrary sixth event, a counting of sign restrictions shows there to be at least 294 additional unrealizable sets of permutations, also by a computer search. As a concluding comment, given the minimal non-realizable sets that have been shown here for $1 + 1$ relativity and for $3 + 1$ relativity, it might be reasonable to conjecture that in $n + 1$ relativity, there is exactly one non-realizable set of $n + 1$ events for $n + 1$ observers, specifically the set based on the cyclic permutation.

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Appendix 1: Five Observers of Six Events in 3-Dimensional Space: All Cases.

All 35 cases are listed below. The first eight entries are the time invariant sets and the remainder of the listing pairs up each set with its time reversal.

- 1: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (2, 6, 1, 5, 4, 3), (4, 3, 2, 6, 1, 5), (5, 4, 2, 1, 3, 6)\}$
- 2: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (3, 4, 5, 6, 1, 2), (5, 4, 2, 1, 3, 6), (5, 6, 1, 2, 3, 4)\}$
- 3: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 4, 2, 3), (2, 5, 4, 6, 1, 3), (4, 5, 3, 1, 2, 6), (4, 6, 1, 3, 2, 5)\}$
- 4: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 4, 2, 3), (3, 6, 1, 4, 5, 2), (4, 5, 3, 1, 2, 6), (5, 2, 3, 6, 1, 4)\}$ self inverse
- 5: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 4, 1, 2, 3, 5)\}$
- 6: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (4, 3, 5, 1, 6, 2), (5, 1, 6, 2, 4, 3), (6, 4, 1, 2, 3, 5)\}$
- 7: $\{(1, 2, 3, 4, 5, 6), (3, 6, 1, 4, 5, 2), (4, 2, 3, 6, 5, 1), (5, 2, 3, 6, 1, 4), (6, 2, 1, 4, 5, 3)\}$
- 8: $\{(1, 2, 3, 4, 5, 6), (4, 1, 6, 3, 5, 2), (4, 2, 3, 6, 5, 1), (5, 2, 4, 1, 6, 3), (6, 2, 1, 4, 5, 3)\}$
- 9a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (2, 4, 6, 5, 1, 3), (5, 6, 1, 2, 3, 4), (6, 3, 2, 1, 4, 5)\}$
- 9b: $\{(1, 2, 3, 4, 5, 6), (2, 3, 6, 5, 4, 1), (3, 4, 5, 6, 1, 2), (4, 6, 2, 1, 3, 5), (5, 4, 2, 1, 3, 6)\}$
- 10a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 3, 2, 1, 4, 5)\}$
- 10b: $\{(1, 2, 3, 4, 5, 6), (2, 3, 6, 5, 4, 1), (3, 4, 5, 6, 1, 2), (5, 4, 2, 1, 3, 6), (5, 6, 1, 2, 3, 4)\}$
- 11a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (3, 5, 4, 1, 6, 2), (4, 3, 2, 6, 1, 5), (6, 3, 1, 2, 5, 4)\}$
- 11b: $\{(1, 2, 3, 4, 5, 6), (2, 6, 1, 5, 4, 3), (3, 2, 5, 6, 4, 1), (5, 1, 6, 3, 2, 4), (5, 4, 2, 1, 3, 6)\}$
- 12a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (3, 6, 1, 5, 2, 4), (4, 3, 2, 6, 1, 5), (5, 3, 4, 1, 2, 6)\}$
- 12b: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 3, 4, 2), (2, 6, 1, 5, 4, 3), (3, 5, 2, 6, 1, 4), (5, 4, 2, 1, 3, 6)\}$
- 13a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (3, 6, 4, 1, 2, 5), (5, 3, 1, 6, 2, 4), (6, 2, 1, 5, 3, 4)\}$
- 13b: $\{(1, 2, 3, 4, 5, 6), (2, 5, 6, 3, 1, 4), (3, 4, 2, 6, 5, 1), (3, 5, 1, 6, 4, 2), (5, 4, 2, 1, 3, 6)\}$
- 14a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (4, 2, 5, 6, 1, 3), (5, 3, 4, 1, 2, 6), (5, 6, 1, 2, 3, 4)\}$
- 14b: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 3, 4, 2), (3, 4, 5, 6, 1, 2), (4, 6, 1, 2, 5, 3), (5, 4, 2, 1, 3, 6)\}$
- 15a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (4, 2, 6, 3, 1, 5), (5, 2, 4, 1, 6, 3), (6, 2, 1, 5, 3, 4)\}$
- 15b: $\{(1, 2, 3, 4, 5, 6), (2, 6, 4, 1, 5, 3), (3, 4, 2, 6, 5, 1), (4, 1, 6, 3, 5, 2), (5, 4, 2, 1, 3, 6)\}$
- 16a: $\{(1, 2, 3, 4, 5, 6), (1, 4, 6, 5, 3, 2), (5, 2, 4, 1, 6, 3), (5, 3, 1, 6, 2, 4), (6, 2, 1, 5, 3, 4)\}$
- 16b: $\{(1, 2, 3, 4, 5, 6), (3, 4, 2, 6, 5, 1), (3, 5, 1, 6, 4, 2), (4, 1, 6, 3, 5, 2), (5, 4, 2, 1, 3, 6)\}$
- 17a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 4, 6, 3, 2), (2, 5, 6, 1, 4, 3), (4, 5, 2, 3, 1, 6), (4, 6, 2, 1, 3, 5)\}$
- 17b: $\{(1, 2, 3, 4, 5, 6), (1, 6, 4, 5, 2, 3), (2, 4, 6, 5, 1, 3), (4, 3, 6, 1, 2, 5), (5, 4, 1, 3, 2, 6)\}$
- 18a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 4, 6, 3, 2), (2, 5, 6, 1, 4, 3), (4, 6, 2, 1, 3, 5), (6, 3, 1, 2, 5, 4)\}$
- 18b: $\{(1, 2, 3, 4, 5, 6), (2, 4, 6, 5, 1, 3), (3, 2, 5, 6, 4, 1), (4, 3, 6, 1, 2, 5), (5, 4, 1, 3, 2, 6)\}$
- 19a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 4, 6, 3, 2), (3, 4, 5, 6, 1, 2), (4, 6, 2, 1, 3, 5), (6, 3, 1, 2, 5, 4)\}$
- 19b: $\{(1, 2, 3, 4, 5, 6), (2, 4, 6, 5, 1, 3), (3, 2, 5, 6, 4, 1), (5, 4, 1, 3, 2, 6), (5, 6, 1, 2, 3, 4)\}$
- 20a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 4, 6, 3, 2), (3, 4, 5, 6, 1, 2), (5, 3, 2, 1, 6, 4), (6, 3, 1, 2, 5, 4)\}$
- 20b: $\{(1, 2, 3, 4, 5, 6), (3, 1, 6, 5, 4, 2), (3, 2, 5, 6, 4, 1), (5, 4, 1, 3, 2, 6), (5, 6, 1, 2, 3, 4)\}$
- 21a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 3, 4, 2), (2, 5, 6, 1, 4, 3), (3, 6, 2, 1, 5, 4), (5, 3, 2, 4, 1, 6)\}$
- 21b: $\{(1, 2, 3, 4, 5, 6), (1, 6, 3, 5, 4, 2), (3, 2, 6, 5, 1, 4), (4, 3, 6, 1, 2, 5), (5, 3, 4, 1, 2, 6)\}$
- 22a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 3, 4, 2), (2, 5, 6, 1, 4, 3), (3, 6, 2, 1, 5, 4), (6, 4, 1, 2, 3, 5)\}$
- 22b: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (3, 2, 6, 5, 1, 4), (4, 3, 6, 1, 2, 5), (5, 3, 4, 1, 2, 6)\}$

- 23a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 3, 4, 2), (2, 5, 6, 1, 4, 3), (4, 6, 1, 2, 5, 3), (6, 3, 2, 1, 4, 5)\}$
 23b: $\{(1, 2, 3, 4, 5, 6), (2, 3, 6, 5, 4, 1), (4, 2, 5, 6, 1, 3), (4, 3, 6, 1, 2, 5), (5, 3, 4, 1, 2, 6)\}$
 24a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 3, 4, 2), (3, 4, 5, 6, 1, 2), (4, 5, 2, 1, 6, 3), (6, 4, 1, 2, 3, 5)\}$
 24b: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (4, 1, 6, 5, 2, 3), (5, 3, 4, 1, 2, 6), (5, 6, 1, 2, 3, 4)\}$
 25a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 4, 2, 3), (2, 5, 4, 6, 1, 3), (4, 6, 1, 3, 2, 5), (6, 2, 3, 1, 5, 4)\}$
 25b: $\{(1, 2, 3, 4, 5, 6), (2, 5, 4, 6, 1, 3), (3, 2, 6, 4, 5, 1), (4, 5, 3, 1, 2, 6), (4, 6, 1, 3, 2, 5)\}$
 26a: $\{(1, 2, 3, 4, 5, 6), (1, 5, 6, 4, 2, 3), (2, 5, 4, 6, 1, 3), (5, 3, 2, 1, 6, 4), (6, 2, 3, 1, 5, 4)\}$
 26b: $\{(1, 2, 3, 4, 5, 6), (3, 1, 6, 5, 4, 2), (3, 2, 6, 4, 5, 1), (4, 5, 3, 1, 2, 6), (4, 6, 1, 3, 2, 5)\}$
 27a: $\{(1, 2, 3, 4, 5, 6), (1, 6, 3, 5, 4, 2), (2, 6, 1, 5, 4, 3), (3, 6, 2, 4, 1, 5), (5, 3, 2, 1, 6, 4)\}$
 27b: $\{(1, 2, 3, 4, 5, 6), (2, 6, 3, 5, 1, 4), (3, 1, 6, 5, 4, 2), (4, 3, 2, 6, 1, 5), (5, 3, 2, 4, 1, 6)\}$
 28a: $\{(1, 2, 3, 4, 5, 6), (1, 6, 3, 5, 4, 2), (2, 6, 5, 1, 3, 4), (3, 6, 2, 4, 1, 5), (4, 6, 1, 2, 5, 3)\}$
 28b: $\{(1, 2, 3, 4, 5, 6), (2, 6, 3, 5, 1, 4), (3, 4, 6, 2, 1, 5), (4, 2, 5, 6, 1, 3), (5, 3, 2, 4, 1, 6)\}$
 29a: $\{(1, 2, 3, 4, 5, 6), (2, 3, 6, 5, 4, 1), (3, 1, 6, 5, 4, 2), (5, 6, 1, 2, 3, 4), (6, 4, 1, 2, 3, 5)\}$
 29b: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (3, 4, 5, 6, 1, 2), (5, 3, 2, 1, 6, 4), (6, 3, 2, 1, 4, 5)\}$
 30a: $\{(1, 2, 3, 4, 5, 6), (2, 3, 6, 5, 4, 1), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 4, 1, 2, 3, 5)\}$
 30b: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (3, 4, 5, 6, 1, 2), (5, 6, 1, 2, 3, 4), (6, 3, 2, 1, 4, 5)\}$
 31a: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (3, 6, 2, 4, 1, 5), (4, 3, 5, 1, 6, 2), (5, 3, 2, 1, 6, 4)\}$
 31b: $\{(1, 2, 3, 4, 5, 6), (2, 6, 3, 5, 1, 4), (3, 1, 6, 5, 4, 2), (5, 1, 6, 2, 4, 3), (6, 4, 1, 2, 3, 5)\}$
 32a: $\{(1, 2, 3, 4, 5, 6), (2, 4, 5, 6, 3, 1), (4, 6, 1, 3, 2, 5), (5, 1, 4, 6, 2, 3), (6, 2, 1, 5, 3, 4)\}$
 32b: $\{(1, 2, 3, 4, 5, 6), (2, 5, 4, 6, 1, 3), (3, 4, 2, 6, 5, 1), (4, 5, 1, 3, 6, 2), (6, 4, 1, 2, 3, 5)\}$
 33a: $\{(1, 2, 3, 4, 5, 6), (2, 4, 6, 3, 5, 1), (3, 6, 2, 1, 5, 4), (4, 3, 1, 6, 5, 2), (6, 1, 4, 2, 5, 3)\}$ group of order 5
 33b: $\{(1, 2, 3, 4, 5, 6), (3, 2, 6, 5, 1, 4), (4, 2, 5, 3, 6, 1), (5, 2, 1, 6, 4, 3), (6, 2, 4, 1, 3, 5)\}$ group of order 5
 34a: $\{(1, 2, 3, 4, 5, 6), (2, 4, 6, 3, 5, 1), (3, 6, 4, 1, 2, 5), (4, 1, 6, 5, 2, 3), (6, 2, 1, 5, 3, 4)\}$
 34b: $\{(1, 2, 3, 4, 5, 6), (2, 5, 6, 3, 1, 4), (3, 4, 2, 6, 5, 1), (4, 5, 2, 1, 6, 3), (6, 2, 4, 1, 3, 5)\}$
 35a: $\{(1, 2, 3, 4, 5, 6), (2, 6, 4, 1, 5, 3), (3, 4, 2, 6, 5, 1), (4, 1, 6, 3, 5, 2), (6, 3, 1, 2, 5, 4)\}$ group of order 5
 35b: $\{(1, 2, 3, 4, 5, 6), (3, 2, 5, 6, 4, 1), (4, 2, 6, 3, 1, 5), (5, 2, 4, 1, 6, 3), (6, 2, 1, 5, 3, 4)\}$ group of order 5

Appendix 2: Calculation of α_i 's and β .

For a 4×3 matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}$$

consider the four 3×3 matrices obtained by deleting a single row

$$\begin{aligned} A_1 &= \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} & A_2 &= \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ A_3 &= \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} & A_4 &= \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{aligned}$$

There is the following general linear relation between the rows of A

$$\begin{aligned} 0 &= \det(A_1) \cdot (a_{1,1}, a_{1,2}, a_{1,3}) - \det(A_2) \cdot (a_{2,1}, a_{2,2}, a_{2,3}) \\ &\quad + \det(A_3) \cdot (a_{3,1}, a_{3,2}, a_{3,3}) - \det(A_4) \cdot (a_{4,1}, a_{4,2}, a_{4,3}) \end{aligned}$$

Writing $D_i = \det(A_i)$ for $i = 1, 2, 3, 4$, and setting

$$\begin{aligned} \tilde{a}_1 &= (a_{1,1}, a_{1,2}, a_{1,3}) \\ \tilde{a}_2 &= (a_{2,1}, a_{2,2}, a_{2,3}) \\ \tilde{a}_3 &= (a_{3,1}, a_{3,2}, a_{3,3}) \\ \tilde{a}_4 &= (a_{4,1}, a_{4,2}, a_{4,3}) \end{aligned}$$

so that if $D_4 \neq 0$ then

$$\tilde{a}_4 = \alpha_1 \tilde{a}_1 + \alpha_2 \tilde{a}_2 + \alpha_3 \tilde{a}_3$$

with

$$\alpha_1 = D_1/D_4 \quad \alpha_2 = -D_2/D_4 \quad \alpha_3 = D_3/D_4$$

For the set

$$Q_0 = \{(1, 2, 3, 4, 5), (2, 3, 4, 5, 1), (3, 4, 5, 1, 2), (4, 5, 1, 2, 3), (5, 1, 2, 3, 4)\}$$

the 4×3 matrix is

$$A = \begin{pmatrix} h_4^{(1)} + h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} & h_3^{(2)} + h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} & h_2^{(3)} + h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} \\ -h_1^{(1)} - h_2^{(1)} - h_3^{(1)} - h_4^{(1)} - h_1^{(0)} & h_4^{(2)} - h_1^{(0)} & h_3^{(3)} - h_1^{(0)} \\ h_1^{(1)} - h_2^{(0)} & -h_1^{(2)} - h_2^{(2)} - h_3^{(2)} - h_4^{(2)} - h_2^{(0)} & h_4^{(3)} - h_2^{(0)} \\ h_2^{(1)} - h_3^{(0)} & h_1^{(2)} - h_3^{(0)} & -h_1^{(3)} - h_2^{(3)} - h_3^{(3)} - h_4^{(3)} - h_3^{(0)} \end{pmatrix}$$

and

$$\begin{aligned} b_1 &= -h_1^{(0)} - h_2^{(0)} - h_3^{(0)} - h_4^{(0)} \\ b_2 &= h_1^{(0)} \\ b_3 &= h_2^{(0)} \\ b_4 &= h_3^{(0)} \end{aligned}$$

Writing $D_i = \det(A_i)$ for $i = 1, 2, 3, 4$, the computations below show that $D_1, D_3 < 0$ and $D_2, D_4 > 0$. Therefore $\alpha_1, \alpha_2, \alpha_3 < 0$. Also

$$\beta = b_4 - \alpha_1 b_1 - \alpha_2 b_2 - \alpha_3 b_3$$

and a computation of β is also given below, which shows that $\beta < 0$. Putting all this together demonstrates that the permutation set Q_0 is not realizable.

The following results were obtained by the computer algebra system magma:

$$A_4^T = \begin{pmatrix} h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} + h_4^{(1)} & h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_3^{(2)} + h_4^{(0)} & h_1^{(0)} + h_2^{(0)} + h_2^{(3)} + h_3^{(0)} + h_4^{(0)} \\ -h_1^{(0)} - h_1^{(1)} - h_2^{(1)} - h_3^{(1)} - h_4^{(1)} & -h_1^{(0)} + h_4^{(2)} & -h_1^{(0)} + h_3^{(3)} \\ h_1^{(1)} - h_2^{(0)} & -h_1^{(2)} - h_2^{(0)} - h_2^{(2)} - h_3^{(2)} - h_4^{(2)} & -h_2^{(0)} + h_4^{(3)} \end{pmatrix}$$

$$D_4 = \det(A_4)$$

$$\begin{aligned} &= h_1^{(0)} h_1^{(1)} h_1^{(2)} + h_1^{(0)} h_1^{(1)} h_2^{(2)} + h_1^{(0)} h_1^{(1)} h_2^{(3)} + h_1^{(0)} h_1^{(1)} h_3^{(3)} + h_1^{(0)} h_1^{(1)} h_4^{(3)} + h_1^{(0)} h_1^{(2)} h_2^{(1)} \\ &+ h_1^{(0)} h_1^{(2)} h_2^{(3)} + h_1^{(0)} h_1^{(2)} h_3^{(3)} + h_1^{(0)} h_1^{(2)} h_4^{(3)} + h_1^{(0)} h_2^{(1)} h_2^{(2)} + h_1^{(0)} h_2^{(1)} h_3^{(2)} + h_1^{(0)} h_2^{(1)} h_4^{(2)} \\ &+ h_1^{(0)} h_2^{(1)} h_4^{(3)} + h_1^{(0)} h_2^{(2)} h_2^{(3)} + h_1^{(0)} h_2^{(2)} h_3^{(3)} + h_1^{(0)} h_2^{(2)} h_4^{(3)} + h_1^{(0)} h_2^{(3)} h_3^{(2)} + h_1^{(0)} h_2^{(3)} h_4^{(2)} \\ &+ h_1^{(0)} h_3^{(1)} h_2^{(2)} + h_1^{(0)} h_3^{(1)} h_4^{(2)} + h_1^{(0)} h_3^{(1)} h_4^{(3)} + h_1^{(0)} h_3^{(2)} h_3^{(3)} + h_1^{(0)} h_3^{(2)} h_4^{(3)} + h_1^{(0)} h_3^{(3)} h_4^{(2)} \\ &+ h_1^{(0)} h_4^{(2)} h_3^{(3)} + h_1^{(1)} h_1^{(2)} h_2^{(0)} + h_1^{(1)} h_1^{(2)} h_2^{(3)} + h_1^{(1)} h_1^{(2)} h_3^{(0)} + h_1^{(1)} h_1^{(2)} h_4^{(0)} + h_1^{(1)} h_2^{(0)} h_2^{(2)} \\ &+ h_1^{(1)} h_2^{(0)} h_2^{(3)} + h_1^{(1)} h_2^{(0)} h_3^{(3)} + h_1^{(1)} h_2^{(0)} h_4^{(3)} + h_1^{(1)} h_2^{(2)} h_2^{(3)} + h_1^{(1)} h_2^{(2)} h_3^{(3)} + h_1^{(1)} h_2^{(2)} h_4^{(3)} \\ &+ h_1^{(1)} h_2^{(3)} h_3^{(2)} + h_1^{(1)} h_2^{(3)} h_4^{(2)} + h_1^{(1)} h_3^{(0)} h_3^{(3)} + h_1^{(1)} h_3^{(0)} h_4^{(3)} + h_1^{(1)} h_3^{(1)} h_3^{(2)} + h_1^{(1)} h_3^{(1)} h_4^{(2)} \\ &+ h_1^{(1)} h_3^{(1)} h_4^{(3)} + h_1^{(1)} h_3^{(2)} h_4^{(0)} + h_1^{(1)} h_4^{(0)} h_4^{(3)} + h_1^{(1)} h_4^{(2)} h_2^{(1)} + h_1^{(1)} h_4^{(2)} h_3^{(1)} + h_1^{(1)} h_4^{(2)} h_4^{(3)} \\ &+ h_1^{(1)} h_4^{(3)} h_3^{(1)} + h_1^{(2)} h_2^{(1)} h_2^{(3)} + h_1^{(2)} h_2^{(1)} h_3^{(0)} + h_1^{(2)} h_2^{(1)} h_4^{(0)} + h_1^{(2)} h_2^{(3)} h_3^{(1)} + h_1^{(2)} h_2^{(3)} h_4^{(1)} \\ &+ h_1^{(2)} h_3^{(0)} h_3^{(1)} + h_1^{(2)} h_3^{(0)} h_4^{(1)} + h_1^{(2)} h_3^{(0)} h_4^{(2)} + h_1^{(2)} h_3^{(0)} h_4^{(3)} + h_1^{(2)} h_3^{(1)} h_4^{(2)} + h_1^{(2)} h_3^{(1)} h_4^{(3)} \\ &+ h_1^{(2)} h_4^{(0)} h_4^{(1)} + h_2^{(0)} h_2^{(1)} h_2^{(2)} + h_2^{(0)} h_2^{(1)} h_2^{(3)} + h_2^{(0)} h_2^{(1)} h_4^{(2)} + h_2^{(0)} h_2^{(1)} h_4^{(3)} + h_2^{(0)} h_2^{(2)} h_3^{(1)} \\ &+ h_2^{(0)} h_2^{(2)} h_3^{(3)} + h_2^{(0)} h_2^{(2)} h_4^{(1)} + h_2^{(0)} h_2^{(2)} h_4^{(3)} + h_2^{(0)} h_2^{(3)} h_3^{(1)} + h_2^{(0)} h_2^{(3)} h_4^{(1)} + h_2^{(0)} h_2^{(3)} h_4^{(2)} \\ &+ h_2^{(0)} h_3^{(1)} h_4^{(3)} + h_2^{(0)} h_3^{(3)} h_4^{(1)} + h_2^{(0)} h_3^{(3)} h_4^{(2)} + h_2^{(0)} h_4^{(1)} h_4^{(3)} + h_2^{(0)} h_4^{(2)} h_4^{(3)} + h_2^{(0)} h_4^{(3)} h_2^{(2)} \\ &+ h_2^{(1)} h_2^{(2)} h_3^{(0)} + h_2^{(1)} h_2^{(2)} h_4^{(0)} + h_2^{(1)} h_2^{(3)} h_3^{(2)} + h_2^{(1)} h_2^{(3)} h_4^{(2)} + h_2^{(1)} h_3^{(0)} h_3^{(2)} + h_2^{(1)} h_3^{(0)} h_4^{(2)} \\ &+ h_2^{(1)} h_3^{(0)} h_4^{(3)} + h_2^{(1)} h_3^{(2)} h_4^{(0)} + h_2^{(1)} h_3^{(2)} h_4^{(3)} + h_2^{(1)} h_4^{(0)} h_4^{(2)} + h_2^{(1)} h_4^{(0)} h_4^{(3)} + h_2^{(1)} h_4^{(2)} h_3^{(1)} \\ &+ h_2^{(1)} h_4^{(2)} h_3^{(3)} + h_2^{(1)} h_4^{(3)} h_3^{(1)} + h_2^{(1)} h_4^{(3)} h_4^{(1)} + h_2^{(2)} h_3^{(0)} h_3^{(1)} + h_2^{(2)} h_3^{(0)} h_4^{(1)} + h_2^{(2)} h_3^{(0)} h_4^{(2)} \\ &+ h_2^{(2)} h_3^{(0)} h_4^{(3)} + h_2^{(2)} h_3^{(1)} h_4^{(2)} + h_2^{(2)} h_3^{(1)} h_4^{(3)} + h_2^{(2)} h_3^{(2)} h_4^{(1)} + h_2^{(2)} h_3^{(2)} h_4^{(2)} + h_2^{(2)} h_3^{(2)} h_4^{(3)} \\ &+ h_2^{(2)} h_3^{(3)} h_4^{(1)} + h_2^{(2)} h_4^{(1)} h_4^{(2)} + h_2^{(2)} h_4^{(1)} h_4^{(3)} + h_2^{(2)} h_4^{(2)} h_3^{(1)} + h_2^{(2)} h_4^{(2)} h_3^{(3)} + h_2^{(2)} h_4^{(2)} h_4^{(1)} \\ &+ h_2^{(2)} h_4^{(2)} h_4^{(2)} + h_2^{(2)} h_4^{(3)} h_3^{(1)} + h_2^{(2)} h_4^{(3)} h_4^{(1)} + h_2^{(2)} h_4^{(3)} h_4^{(2)} + h_2^{(3)} h_3^{(1)} h_4^{(2)} + h_2^{(3)} h_3^{(1)} h_4^{(3)} \\ &+ h_2^{(3)} h_3^{(2)} h_4^{(1)} + h_2^{(3)} h_3^{(2)} h_4^{(2)} + h_2^{(3)} h_3^{(2)} h_4^{(3)} + h_2^{(3)} h_4^{(1)} h_4^{(2)} + h_2^{(3)} h_4^{(1)} h_4^{(3)} + h_2^{(3)} h_4^{(2)} h_4^{(1)} \\ &+ h_2^{(3)} h_4^{(2)} h_4^{(2)} + h_2^{(3)} h_4^{(2)} h_4^{(3)} + h_2^{(3)} h_4^{(3)} h_4^{(1)} + h_2^{(3)} h_4^{(3)} h_4^{(2)} + h_2^{(3)} h_4^{(3)} h_4^{(3)} \end{aligned}$$

There are 125 terms in the above expression, all of which are positive. Therefore $D_4 > 0$ since $h_i^{(j)} > 0$ for all i and j .

$$A_3^T = \begin{pmatrix} h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} + h_4^{(1)} & -h_1^{(0)} - h_1^{(1)} - h_2^{(1)} - h_3^{(1)} - h_4^{(1)} & h_2^{(1)} - h_3^{(0)} \\ h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_3^{(2)} + h_4^{(0)} & -h_1^{(0)} + h_4^{(2)} & h_1^{(2)} - h_3^{(0)} \\ h_1^{(0)} + h_2^{(0)} + h_2^{(3)} + h_3^{(0)} + h_4^{(0)} & -h_1^{(0)} + h_3^{(3)} & -h_1^{(3)} - h_2^{(3)} - h_3^{(0)} - h_3^{(3)} - h_4^{(3)} \end{pmatrix}$$

$$D_3 = \det(A_3)$$

$$\begin{aligned} &= -h_1^{(0)} h_1^{(1)} h_1^{(2)} - h_1^{(0)} h_1^{(1)} h_1^{(3)} - h_1^{(0)} h_1^{(1)} h_2^{(3)} - h_1^{(0)} h_1^{(1)} h_3^{(3)} - h_1^{(0)} h_1^{(1)} h_4^{(3)} - h_1^{(0)} h_1^{(2)} h_2^{(1)} \\ &\quad - h_1^{(0)} h_1^{(2)} h_2^{(3)} - h_1^{(0)} h_1^{(2)} h_3^{(1)} - h_1^{(0)} h_1^{(2)} h_3^{(3)} - h_1^{(0)} h_1^{(3)} h_2^{(1)} - h_1^{(0)} h_1^{(3)} h_3^{(1)} - h_1^{(0)} h_1^{(3)} h_3^{(2)} \\ &\quad - h_1^{(0)} h_1^{(3)} h_4^{(2)} - h_1^{(0)} h_2^{(1)} h_3^{(2)} - h_1^{(0)} h_2^{(1)} h_4^{(2)} - h_1^{(0)} h_2^{(1)} h_4^{(3)} - h_1^{(0)} h_2^{(3)} h_3^{(1)} - h_1^{(0)} h_2^{(3)} h_3^{(2)} \\ &\quad - h_1^{(0)} h_2^{(3)} h_4^{(2)} - h_1^{(0)} h_3^{(1)} h_3^{(3)} - h_1^{(0)} h_3^{(1)} h_4^{(3)} - h_1^{(0)} h_3^{(2)} h_3^{(3)} - h_1^{(0)} h_3^{(2)} h_4^{(3)} - h_1^{(0)} h_3^{(3)} h_4^{(2)} \\ &\quad - h_1^{(0)} h_4^{(2)} h_4^{(3)} - h_1^{(1)} h_1^{(2)} h_2^{(0)} - h_1^{(1)} h_1^{(2)} h_2^{(3)} - h_1^{(1)} h_1^{(2)} h_3^{(0)} - h_1^{(1)} h_1^{(2)} h_4^{(0)} - h_1^{(1)} h_1^{(3)} h_2^{(0)} \\ &\quad - h_1^{(1)} h_1^{(3)} h_3^{(0)} - h_1^{(1)} h_1^{(3)} h_3^{(2)} - h_1^{(1)} h_1^{(3)} h_4^{(0)} - h_1^{(1)} h_2^{(0)} h_2^{(3)} - h_1^{(1)} h_2^{(0)} h_3^{(3)} - h_1^{(1)} h_2^{(0)} h_4^{(3)} \\ &\quad - h_1^{(1)} h_2^{(3)} h_3^{(2)} - h_1^{(1)} h_2^{(3)} h_4^{(0)} - h_1^{(1)} h_3^{(0)} h_2^{(2)} - h_1^{(1)} h_3^{(0)} h_3^{(3)} - h_1^{(1)} h_3^{(0)} h_4^{(3)} - h_1^{(1)} h_3^{(2)} h_3^{(3)} \\ &\quad - h_1^{(1)} h_3^{(2)} h_4^{(3)} - h_1^{(1)} h_3^{(3)} h_4^{(0)} - h_1^{(1)} h_4^{(0)} h_4^{(3)} - h_1^{(2)} h_2^{(0)} h_2^{(1)} - h_1^{(2)} h_2^{(0)} h_3^{(1)} - h_1^{(2)} h_2^{(0)} h_3^{(2)} \\ &\quad - h_1^{(2)} h_2^{(0)} h_4^{(1)} - h_1^{(2)} h_2^{(1)} h_2^{(2)} - h_1^{(2)} h_2^{(1)} h_3^{(2)} - h_1^{(2)} h_2^{(1)} h_4^{(0)} - h_1^{(2)} h_2^{(3)} h_3^{(1)} - h_1^{(2)} h_2^{(3)} h_4^{(1)} \\ &\quad - h_1^{(2)} h_3^{(0)} h_3^{(1)} - h_1^{(2)} h_3^{(0)} h_3^{(3)} - h_1^{(2)} h_3^{(0)} h_4^{(1)} - h_1^{(2)} h_3^{(1)} h_4^{(0)} - h_1^{(2)} h_3^{(1)} h_4^{(2)} - h_1^{(2)} h_3^{(3)} h_4^{(1)} \\ &\quad - h_1^{(2)} h_4^{(0)} h_4^{(1)} - h_1^{(3)} h_2^{(0)} h_2^{(1)} - h_1^{(3)} h_2^{(0)} h_3^{(1)} - h_1^{(3)} h_2^{(0)} h_4^{(1)} - h_1^{(3)} h_2^{(1)} h_4^{(2)} - h_1^{(3)} h_2^{(1)} h_3^{(0)} \\ &\quad - h_1^{(3)} h_2^{(1)} h_3^{(2)} - h_1^{(3)} h_2^{(1)} h_4^{(0)} - h_1^{(3)} h_3^{(0)} h_3^{(1)} - h_1^{(3)} h_3^{(0)} h_4^{(1)} - h_1^{(3)} h_3^{(1)} h_4^{(2)} - h_1^{(3)} h_3^{(1)} h_3^{(2)} \\ &\quad - h_1^{(3)} h_3^{(1)} h_4^{(0)} - h_1^{(3)} h_3^{(2)} h_4^{(1)} - h_1^{(3)} h_4^{(0)} h_4^{(1)} - h_1^{(3)} h_4^{(0)} h_4^{(2)} - h_1^{(3)} h_4^{(1)} h_4^{(2)} - h_2^{(0)} h_2^{(1)} h_2^{(3)} \\ &\quad - h_2^{(0)} h_2^{(1)} h_4^{(2)} - h_2^{(0)} h_2^{(1)} h_4^{(3)} - h_2^{(0)} h_2^{(3)} h_3^{(1)} - h_2^{(0)} h_2^{(3)} h_4^{(1)} - h_2^{(0)} h_2^{(3)} h_4^{(2)} - h_2^{(0)} h_3^{(1)} h_3^{(3)} \\ &\quad - h_2^{(0)} h_3^{(1)} h_4^{(3)} - h_2^{(0)} h_3^{(3)} h_4^{(1)} - h_2^{(0)} h_3^{(3)} h_4^{(2)} - h_2^{(0)} h_4^{(1)} h_4^{(3)} - h_2^{(0)} h_4^{(2)} h_4^{(3)} - h_2^{(1)} h_2^{(3)} h_3^{(2)} \\ &\quad - h_2^{(1)} h_2^{(3)} h_4^{(0)} - h_2^{(1)} h_2^{(3)} h_4^{(2)} - h_2^{(1)} h_3^{(0)} h_2^{(2)} - h_2^{(1)} h_3^{(0)} h_4^{(2)} - h_2^{(1)} h_3^{(0)} h_4^{(3)} - h_2^{(1)} h_3^{(2)} h_4^{(3)} \\ &\quad - h_2^{(1)} h_4^{(0)} h_4^{(2)} - h_2^{(1)} h_4^{(0)} h_4^{(3)} - h_2^{(3)} h_3^{(1)} h_2^{(2)} - h_2^{(3)} h_3^{(1)} h_4^{(0)} - h_2^{(3)} h_3^{(2)} h_4^{(1)} - h_2^{(3)} h_4^{(0)} h_4^{(1)} \\ &\quad - h_2^{(3)} h_4^{(0)} h_4^{(2)} - h_2^{(3)} h_4^{(1)} h_4^{(2)} - h_3^{(0)} h_3^{(1)} h_3^{(2)} - h_3^{(0)} h_3^{(1)} h_3^{(3)} - h_3^{(0)} h_3^{(1)} h_4^{(3)} - h_3^{(0)} h_3^{(2)} h_3^{(3)} \\ &\quad - h_3^{(0)} h_3^{(2)} h_4^{(1)} - h_3^{(0)} h_3^{(3)} h_4^{(2)} - h_3^{(0)} h_4^{(1)} h_4^{(2)} - h_3^{(0)} h_4^{(1)} h_4^{(3)} - h_3^{(0)} h_4^{(2)} h_4^{(3)} - h_3^{(1)} h_3^{(2)} h_3^{(3)} \\ &\quad - h_3^{(1)} h_3^{(2)} h_4^{(3)} - h_3^{(1)} h_3^{(3)} h_4^{(0)} - h_3^{(1)} h_4^{(0)} h_4^{(3)} - h_3^{(2)} h_3^{(3)} h_4^{(1)} - h_3^{(2)} h_4^{(1)} h_4^{(3)} - h_3^{(3)} h_4^{(0)} h_4^{(1)} \\ &\quad - h_3^{(3)} h_4^{(0)} h_4^{(2)} - h_3^{(3)} h_4^{(1)} h_4^{(2)} - h_4^{(0)} h_4^{(1)} h_4^{(3)} - h_4^{(0)} h_4^{(2)} h_4^{(3)} - h_4^{(1)} h_4^{(2)} h_4^{(3)} \end{aligned}$$

There are 125 terms in the above expression, all of which are negative. Therefore $D_3 < 0$ since $h_i^{(j)} > 0$ for all i and j .

$$A_2^r = \begin{pmatrix} h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_4^{(0)} + h_4^{(1)} & h_1^{(1)} - h_2^{(0)} & h_2^{(1)} - h_3^{(0)} \\ h_1^{(0)} + h_2^{(0)} + h_3^{(0)} + h_3^{(2)} + h_4^{(0)} & -h_1^{(2)} - h_2^{(0)} - h_2^{(2)} - h_3^{(2)} - h_4^{(2)} & h_1^{(2)} - h_3^{(0)} \\ h_1^{(0)} + h_2^{(0)} + h_2^{(3)} + h_3^{(0)} + h_4^{(0)} & -h_2^{(0)} + h_4^{(3)} & -h_1^{(3)} - h_2^{(3)} - h_3^{(0)} - h_3^{(3)} - h_4^{(3)} \end{pmatrix}$$

$$D_2 = \det(A_2)$$

$$\begin{aligned} &= h_1^{(0)} h_1^{(1)} h_2^{(1)} + h_1^{(0)} h_1^{(1)} h_1^{(3)} + h_1^{(0)} h_1^{(1)} h_2^{(3)} + h_1^{(0)} h_1^{(1)} h_3^{(3)} + h_1^{(0)} h_1^{(1)} h_4^{(3)} + h_1^{(0)} h_1^{(2)} h_1^{(3)} \\ &+ h_1^{(0)} h_1^{(2)} h_2^{(1)} + h_1^{(0)} h_1^{(2)} h_2^{(3)} + h_1^{(0)} h_1^{(2)} h_3^{(3)} + h_1^{(0)} h_1^{(2)} h_4^{(3)} + h_1^{(0)} h_1^{(3)} h_2^{(2)} + h_1^{(0)} h_1^{(3)} h_3^{(2)} + h_1^{(0)} h_1^{(3)} h_4^{(2)} \\ &+ h_1^{(0)} h_2^{(1)} h_2^{(2)} + h_1^{(0)} h_2^{(1)} h_2^{(3)} + h_1^{(0)} h_2^{(1)} h_4^{(2)} + h_1^{(0)} h_2^{(1)} h_4^{(3)} + h_1^{(0)} h_2^{(2)} h_2^{(3)} + h_1^{(0)} h_2^{(2)} h_3^{(3)} \\ &+ h_1^{(0)} h_2^{(2)} h_4^{(3)} + h_1^{(0)} h_2^{(3)} h_2^{(3)} + h_1^{(0)} h_2^{(3)} h_4^{(2)} + h_1^{(0)} h_2^{(3)} h_4^{(3)} + h_1^{(0)} h_3^{(2)} h_4^{(3)} + h_1^{(0)} h_3^{(3)} h_4^{(2)} \\ &+ h_1^{(0)} h_4^{(2)} h_4^{(3)} + h_1^{(1)} h_1^{(2)} h_2^{(0)} + h_1^{(1)} h_1^{(2)} h_2^{(3)} + h_1^{(1)} h_1^{(2)} h_3^{(0)} + h_1^{(1)} h_1^{(2)} h_4^{(0)} + h_1^{(1)} h_1^{(3)} h_2^{(0)} \\ &+ h_1^{(1)} h_1^{(3)} h_3^{(0)} + h_1^{(1)} h_1^{(3)} h_3^{(2)} + h_1^{(1)} h_1^{(3)} h_4^{(0)} + h_1^{(1)} h_2^{(0)} h_2^{(3)} + h_1^{(1)} h_2^{(0)} h_3^{(3)} + h_1^{(1)} h_2^{(0)} h_4^{(3)} \\ &+ h_1^{(1)} h_2^{(3)} h_3^{(2)} + h_1^{(1)} h_2^{(3)} h_4^{(0)} + h_1^{(1)} h_3^{(0)} h_3^{(2)} + h_1^{(1)} h_3^{(0)} h_3^{(3)} + h_1^{(1)} h_3^{(0)} h_4^{(3)} + h_1^{(1)} h_3^{(2)} h_3^{(3)} \\ &+ h_1^{(1)} h_3^{(2)} h_4^{(3)} + h_1^{(1)} h_3^{(3)} h_4^{(0)} + h_1^{(1)} h_4^{(0)} h_4^{(3)} + h_1^{(1)} h_4^{(2)} h_3^{(3)} h_2^{(0)} + h_1^{(1)} h_4^{(2)} h_3^{(0)} + h_1^{(1)} h_4^{(2)} h_3^{(3)} h_4^{(0)} \\ &+ h_1^{(2)} h_1^{(3)} h_4^{(1)} + h_1^{(2)} h_2^{(0)} h_2^{(1)} + h_1^{(2)} h_2^{(0)} h_3^{(3)} + h_1^{(2)} h_2^{(0)} h_4^{(1)} + h_1^{(2)} h_2^{(1)} h_2^{(3)} + h_1^{(2)} h_2^{(1)} h_3^{(0)} \\ &+ h_1^{(2)} h_2^{(1)} h_4^{(0)} + h_1^{(2)} h_2^{(2)} h_2^{(3)} h_4^{(0)} + h_1^{(2)} h_2^{(2)} h_4^{(1)} + h_1^{(2)} h_3^{(0)} h_3^{(3)} + h_1^{(2)} h_3^{(0)} h_4^{(1)} + h_1^{(2)} h_3^{(1)} h_3^{(3)} h_4^{(0)} \\ &+ h_1^{(2)} h_3^{(3)} h_4^{(1)} + h_1^{(3)} h_2^{(0)} h_2^{(2)} + h_1^{(3)} h_2^{(0)} h_4^{(1)} + h_1^{(3)} h_2^{(0)} h_4^{(2)} + h_1^{(3)} h_2^{(2)} h_3^{(0)} + h_1^{(3)} h_2^{(2)} h_4^{(0)} \\ &+ h_1^{(3)} h_2^{(2)} h_4^{(1)} + h_1^{(3)} h_3^{(0)} h_3^{(2)} + h_1^{(3)} h_3^{(0)} h_4^{(2)} + h_1^{(3)} h_3^{(2)} h_4^{(0)} + h_1^{(3)} h_3^{(2)} h_4^{(1)} + h_1^{(3)} h_4^{(0)} h_4^{(2)} \\ &+ h_1^{(3)} h_4^{(1)} h_4^{(2)} + h_2^{(0)} h_2^{(1)} h_2^{(2)} + h_2^{(0)} h_2^{(1)} h_2^{(3)} + h_2^{(0)} h_2^{(1)} h_4^{(2)} + h_2^{(0)} h_2^{(1)} h_4^{(3)} + h_2^{(0)} h_2^{(2)} h_2^{(3)} \\ &+ h_2^{(0)} h_2^{(2)} h_3^{(3)} + h_2^{(0)} h_2^{(2)} h_4^{(3)} + h_2^{(0)} h_2^{(3)} h_4^{(1)} + h_2^{(0)} h_2^{(3)} h_4^{(2)} + h_2^{(0)} h_3^{(3)} h_4^{(1)} + h_2^{(0)} h_3^{(3)} h_4^{(2)} \\ &+ h_2^{(0)} h_4^{(1)} h_4^{(3)} + h_2^{(0)} h_4^{(2)} h_4^{(3)} + h_2^{(1)} h_2^{(2)} h_2^{(3)} + h_2^{(1)} h_2^{(2)} h_3^{(0)} + h_2^{(1)} h_2^{(2)} h_3^{(3)} + h_2^{(1)} h_2^{(2)} h_4^{(0)} + h_2^{(1)} h_2^{(2)} h_4^{(3)} \\ &+ h_2^{(1)} h_2^{(3)} h_4^{(2)} + h_2^{(1)} h_3^{(0)} h_3^{(2)} + h_2^{(1)} h_3^{(0)} h_4^{(2)} + h_2^{(1)} h_3^{(0)} h_4^{(3)} + h_2^{(1)} h_3^{(2)} h_4^{(0)} + h_2^{(1)} h_3^{(2)} h_4^{(3)} \\ &+ h_2^{(1)} h_4^{(0)} h_4^{(2)} + h_2^{(1)} h_4^{(0)} h_4^{(3)} + h_2^{(2)} h_2^{(3)} h_4^{(0)} + h_2^{(2)} h_2^{(3)} h_4^{(1)} + h_2^{(2)} h_3^{(0)} h_3^{(3)} + h_2^{(2)} h_3^{(0)} h_4^{(1)} \\ &+ h_2^{(2)} h_3^{(0)} h_4^{(2)} + h_2^{(2)} h_3^{(0)} h_4^{(3)} + h_2^{(2)} h_3^{(2)} h_4^{(0)} + h_2^{(2)} h_3^{(2)} h_4^{(1)} + h_2^{(2)} h_3^{(2)} h_4^{(3)} + h_2^{(2)} h_4^{(1)} h_4^{(3)} \\ &+ h_2^{(2)} h_4^{(2)} h_4^{(3)} + h_2^{(3)} h_4^{(1)} h_4^{(2)} + h_2^{(3)} h_4^{(1)} h_4^{(3)} + h_2^{(3)} h_4^{(2)} h_4^{(3)} + h_3^{(0)} h_3^{(2)} h_3^{(3)} + h_3^{(0)} h_3^{(2)} h_4^{(1)} + h_3^{(0)} h_3^{(2)} h_4^{(2)} \\ &+ h_3^{(0)} h_4^{(1)} h_4^{(2)} + h_3^{(0)} h_4^{(1)} h_4^{(3)} + h_3^{(0)} h_4^{(2)} h_4^{(3)} + h_3^{(2)} h_3^{(3)} h_4^{(0)} + h_3^{(2)} h_3^{(3)} h_4^{(1)} + h_3^{(2)} h_3^{(3)} h_4^{(2)} \\ &+ h_3^{(2)} h_4^{(1)} h_4^{(3)} + h_3^{(3)} h_4^{(0)} h_4^{(2)} + h_3^{(3)} h_4^{(1)} h_4^{(2)} + h_4^{(0)} h_4^{(2)} h_4^{(3)} + h_4^{(1)} h_4^{(2)} h_4^{(3)} \end{aligned}$$

There are 125 terms in the above expression, all of which are positive. Therefore $D_2 > 0$ since $h_i^{(j)} > 0$ for all i and j .

$$A_1^7 = \begin{pmatrix} -h_1^{(0)} - h_1^{(1)} - h_2^{(1)} - h_3^{(1)} - h_4^{(1)} & -h_1^{(1)} - h_2^{(0)} & h_2^{(1)} - h_3^{(0)} \\ -h_1^{(0)} + h_4^{(2)} & -h_1^{(2)} - h_2^{(0)} - h_2^{(2)} - h_3^{(2)} - h_4^{(2)} & h_1^{(2)} - h_3^{(0)} \\ -h_1^{(0)} + h_3^{(3)} & -h_2^{(0)} + h_4^{(3)} & -h_1^{(3)} - h_2^{(3)} - h_3^{(0)} - h_3^{(3)} - h_4^{(3)} \end{pmatrix}$$

$$D_1 = \det(A_1)$$

$$\begin{aligned} &= -h_1^{(0)} h_1^{(1)} h_1^{(2)} - h_1^{(0)} h_1^{(1)} h_1^{(3)} - h_1^{(0)} h_1^{(1)} h_2^{(3)} - h_1^{(0)} h_1^{(1)} h_3^{(3)} - h_1^{(0)} h_1^{(1)} h_4^{(3)} - h_1^{(0)} h_1^{(2)} h_1^{(3)} \\ &\quad - h_1^{(0)} h_1^{(2)} h_2^{(1)} - h_1^{(0)} h_1^{(2)} h_2^{(3)} - h_1^{(0)} h_1^{(2)} h_3^{(3)} - h_1^{(0)} h_1^{(2)} h_4^{(3)} - h_1^{(0)} h_1^{(3)} h_2^{(2)} - h_1^{(0)} h_1^{(3)} h_3^{(2)} - h_1^{(0)} h_1^{(3)} h_4^{(2)} \\ &\quad - h_1^{(0)} h_2^{(1)} h_2^{(2)} - h_1^{(0)} h_2^{(1)} h_2^{(3)} - h_1^{(0)} h_2^{(1)} h_3^{(2)} - h_1^{(0)} h_2^{(1)} h_4^{(2)} - h_1^{(0)} h_2^{(2)} h_2^{(3)} - h_1^{(0)} h_2^{(2)} h_3^{(3)} \\ &\quad - h_1^{(0)} h_2^{(2)} h_4^{(3)} - h_1^{(0)} h_2^{(3)} h_2^{(3)} - h_1^{(0)} h_2^{(3)} h_3^{(2)} - h_1^{(0)} h_2^{(3)} h_4^{(2)} - h_1^{(0)} h_3^{(2)} h_3^{(3)} - h_1^{(0)} h_3^{(2)} h_4^{(3)} \\ &\quad - h_1^{(0)} h_3^{(3)} h_4^{(3)} - h_1^{(1)} h_1^{(2)} h_1^{(3)} - h_1^{(1)} h_1^{(2)} h_2^{(0)} - h_1^{(1)} h_1^{(2)} h_2^{(3)} - h_1^{(1)} h_1^{(2)} h_3^{(0)} - h_1^{(1)} h_1^{(2)} h_3^{(3)} \\ &\quad - h_1^{(1)} h_1^{(3)} h_2^{(2)} - h_1^{(1)} h_1^{(3)} h_2^{(3)} - h_1^{(1)} h_2^{(0)} h_2^{(3)} - h_1^{(1)} h_2^{(0)} h_3^{(3)} - h_1^{(1)} h_2^{(0)} h_4^{(3)} - h_1^{(1)} h_2^{(2)} h_2^{(3)} \\ &\quad - h_1^{(1)} h_2^{(2)} h_3^{(0)} - h_1^{(1)} h_2^{(2)} h_3^{(3)} - h_1^{(1)} h_2^{(2)} h_4^{(3)} - h_1^{(1)} h_2^{(3)} h_3^{(2)} - h_1^{(1)} h_2^{(3)} h_4^{(2)} - h_1^{(1)} h_3^{(0)} h_3^{(3)} \\ &\quad - h_1^{(1)} h_3^{(0)} h_4^{(3)} - h_1^{(1)} h_3^{(2)} h_3^{(3)} - h_1^{(1)} h_3^{(2)} h_4^{(3)} - h_1^{(2)} h_1^{(3)} h_2^{(1)} - h_1^{(2)} h_1^{(3)} h_3^{(1)} - h_1^{(2)} h_1^{(3)} h_4^{(1)} \\ &\quad - h_1^{(2)} h_2^{(0)} h_2^{(1)} - h_1^{(2)} h_2^{(0)} h_2^{(3)} - h_1^{(2)} h_2^{(0)} h_3^{(3)} - h_1^{(2)} h_2^{(0)} h_4^{(3)} - h_1^{(2)} h_2^{(1)} h_2^{(3)} - h_1^{(2)} h_2^{(1)} h_3^{(0)} \\ &\quad - h_1^{(2)} h_2^{(1)} h_3^{(1)} - h_1^{(2)} h_2^{(1)} h_3^{(3)} - h_1^{(2)} h_2^{(1)} h_4^{(3)} - h_1^{(2)} h_2^{(2)} h_3^{(0)} - h_1^{(2)} h_2^{(2)} h_3^{(3)} - h_1^{(2)} h_2^{(2)} h_4^{(3)} \\ &\quad - h_1^{(2)} h_2^{(2)} h_3^{(1)} - h_1^{(2)} h_2^{(2)} h_3^{(3)} - h_1^{(2)} h_2^{(2)} h_4^{(3)} - h_1^{(2)} h_2^{(3)} h_3^{(0)} - h_1^{(2)} h_2^{(3)} h_3^{(3)} - h_1^{(2)} h_2^{(3)} h_4^{(3)} \\ &\quad - h_1^{(2)} h_3^{(0)} h_3^{(1)} - h_1^{(2)} h_3^{(0)} h_3^{(3)} - h_1^{(2)} h_3^{(0)} h_4^{(3)} - h_1^{(2)} h_3^{(1)} h_2^{(2)} - h_1^{(2)} h_3^{(1)} h_2^{(3)} - h_1^{(2)} h_3^{(1)} h_4^{(2)} \\ &\quad - h_1^{(2)} h_3^{(1)} h_4^{(2)} - h_1^{(2)} h_3^{(1)} h_4^{(3)} - h_1^{(2)} h_3^{(2)} h_2^{(1)} - h_1^{(2)} h_3^{(2)} h_2^{(3)} - h_1^{(2)} h_3^{(2)} h_4^{(2)} - h_1^{(2)} h_3^{(2)} h_4^{(3)} \\ &\quad - h_1^{(2)} h_3^{(2)} h_4^{(1)} - h_1^{(2)} h_3^{(2)} h_4^{(3)} - h_1^{(2)} h_3^{(3)} h_2^{(1)} - h_1^{(2)} h_3^{(3)} h_2^{(3)} - h_1^{(2)} h_3^{(3)} h_4^{(2)} - h_1^{(2)} h_3^{(3)} h_4^{(3)} \\ &\quad - h_1^{(2)} h_4^{(1)} h_4^{(3)} - h_1^{(2)} h_4^{(2)} h_4^{(3)} - h_1^{(2)} h_4^{(2)} h_3^{(0)} - h_1^{(2)} h_4^{(2)} h_3^{(3)} - h_1^{(2)} h_4^{(2)} h_4^{(3)} - h_1^{(2)} h_4^{(3)} h_2^{(1)} \\ &\quad - h_1^{(2)} h_4^{(3)} h_2^{(3)} - h_1^{(2)} h_4^{(3)} h_3^{(1)} - h_1^{(2)} h_4^{(3)} h_3^{(3)} - h_1^{(2)} h_4^{(3)} h_4^{(2)} - h_1^{(2)} h_4^{(3)} h_4^{(3)} \\ &\quad - h_1^{(3)} h_2^{(1)} h_2^{(2)} - h_1^{(3)} h_2^{(1)} h_2^{(3)} - h_1^{(3)} h_2^{(1)} h_3^{(2)} - h_1^{(3)} h_2^{(1)} h_3^{(3)} - h_1^{(3)} h_2^{(1)} h_4^{(2)} - h_1^{(3)} h_2^{(1)} h_4^{(3)} \\ &\quad - h_1^{(3)} h_2^{(2)} h_2^{(3)} - h_1^{(3)} h_2^{(2)} h_3^{(0)} - h_1^{(3)} h_2^{(2)} h_3^{(3)} - h_1^{(3)} h_2^{(2)} h_4^{(3)} - h_1^{(3)} h_2^{(3)} h_2^{(1)} - h_1^{(3)} h_2^{(3)} h_2^{(3)} \\ &\quad - h_1^{(3)} h_2^{(3)} h_3^{(1)} - h_1^{(3)} h_2^{(3)} h_3^{(3)} - h_1^{(3)} h_2^{(3)} h_4^{(2)} - h_1^{(3)} h_2^{(3)} h_4^{(3)} - h_1^{(3)} h_3^{(0)} h_3^{(3)} - h_1^{(3)} h_3^{(0)} h_4^{(3)} \\ &\quad - h_1^{(3)} h_3^{(1)} h_3^{(2)} - h_1^{(3)} h_3^{(1)} h_3^{(3)} - h_1^{(3)} h_3^{(1)} h_4^{(2)} - h_1^{(3)} h_3^{(1)} h_4^{(3)} - h_1^{(3)} h_3^{(2)} h_2^{(1)} - h_1^{(3)} h_3^{(2)} h_2^{(3)} \\ &\quad - h_1^{(3)} h_3^{(2)} h_3^{(1)} - h_1^{(3)} h_3^{(2)} h_3^{(3)} - h_1^{(3)} h_3^{(2)} h_4^{(2)} - h_1^{(3)} h_3^{(2)} h_4^{(3)} - h_1^{(3)} h_3^{(3)} h_2^{(1)} - h_1^{(3)} h_3^{(3)} h_2^{(3)} \\ &\quad - h_1^{(3)} h_3^{(3)} h_3^{(1)} - h_1^{(3)} h_3^{(3)} h_3^{(3)} - h_1^{(3)} h_3^{(3)} h_4^{(2)} - h_1^{(3)} h_3^{(3)} h_4^{(3)} - h_1^{(3)} h_4^{(1)} h_4^{(3)} - h_1^{(3)} h_4^{(1)} h_4^{(3)} \\ &\quad - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} \\ &\quad - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} - h_1^{(3)} h_4^{(2)} h_4^{(3)} \end{aligned}$$

There are 125 terms in the above expression, all of which are negative. Therefore $D_1 < 0$ since $h_i^{(j)} > 0$ for all i and j .

Finally

$$\begin{aligned}
\beta D_4 = & b_4 D_4 - D_1 b_1 + D_2 b_2 - D_3 b_3 \\
= & -h_1^{(0)} h_1^{(1)} h_1^{(2)} h_1^{(3)} - h_1^{(0)} h_1^{(1)} h_1^{(3)} h_2^{(2)} - h_1^{(0)} h_1^{(1)} h_2^{(2)} h_2^{(3)} - h_1^{(0)} h_1^{(1)} h_2^{(2)} h_3^{(3)} - h_1^{(0)} h_1^{(1)} h_2^{(2)} h_4^{(3)} \\
& - h_1^{(0)} h_1^{(2)} h_1^{(3)} h_2^{(1)} - h_1^{(0)} h_1^{(2)} h_1^{(3)} h_3^{(1)} - h_1^{(0)} h_1^{(2)} h_2^{(3)} h_3^{(1)} - h_1^{(0)} h_1^{(2)} h_3^{(1)} h_3^{(3)} - h_1^{(0)} h_1^{(3)} h_2^{(1)} h_2^{(2)} \\
& - h_1^{(0)} h_1^{(3)} h_2^{(1)} h_3^{(2)} - h_1^{(0)} h_1^{(3)} h_2^{(1)} h_4^{(2)} - h_1^{(0)} h_1^{(3)} h_2^{(2)} h_3^{(1)} - h_1^{(0)} h_1^{(3)} h_3^{(1)} h_3^{(2)} - h_1^{(0)} h_1^{(3)} h_3^{(1)} h_4^{(2)} \\
& - h_1^{(0)} h_2^{(1)} h_2^{(2)} h_4^{(3)} - h_1^{(0)} h_2^{(2)} h_2^{(3)} h_3^{(1)} - h_1^{(0)} h_2^{(2)} h_3^{(1)} h_3^{(3)} - h_1^{(0)} h_2^{(2)} h_3^{(1)} h_4^{(3)} - h_1^{(0)} h_2^{(3)} h_3^{(1)} h_3^{(2)} \\
& - h_1^{(0)} h_2^{(3)} h_3^{(1)} h_4^{(2)} - h_1^{(0)} h_3^{(1)} h_2^{(2)} h_3^{(3)} - h_1^{(0)} h_3^{(1)} h_2^{(2)} h_4^{(3)} - h_1^{(0)} h_3^{(1)} h_3^{(2)} h_4^{(3)} - h_1^{(0)} h_3^{(1)} h_4^{(2)} h_4^{(3)} \\
& - h_1^{(1)} h_1^{(2)} h_1^{(3)} h_2^{(0)} - h_1^{(1)} h_1^{(2)} h_1^{(3)} h_3^{(0)} - h_1^{(1)} h_1^{(2)} h_1^{(3)} h_4^{(0)} - h_1^{(1)} h_1^{(2)} h_2^{(3)} h_4^{(0)} - h_1^{(1)} h_1^{(3)} h_2^{(0)} h_2^{(2)} \\
& - h_1^{(1)} h_1^{(3)} h_2^{(2)} h_3^{(0)} - h_1^{(1)} h_1^{(3)} h_2^{(2)} h_4^{(0)} - h_1^{(1)} h_1^{(3)} h_3^{(0)} h_3^{(2)} - h_1^{(1)} h_1^{(3)} h_3^{(2)} h_4^{(0)} - h_1^{(1)} h_2^{(0)} h_2^{(2)} h_2^{(3)} \\
& - h_1^{(1)} h_2^{(0)} h_2^{(2)} h_3^{(3)} - h_1^{(1)} h_2^{(0)} h_2^{(2)} h_4^{(3)} - h_1^{(1)} h_2^{(2)} h_2^{(3)} h_4^{(0)} - h_1^{(1)} h_2^{(2)} h_3^{(0)} h_3^{(3)} - h_1^{(1)} h_2^{(2)} h_3^{(0)} h_4^{(3)} \\
& - h_1^{(1)} h_2^{(2)} h_3^{(3)} h_4^{(0)} - h_1^{(1)} h_2^{(2)} h_4^{(0)} h_4^{(3)} - h_1^{(1)} h_2^{(3)} h_3^{(2)} h_4^{(0)} - h_1^{(1)} h_2^{(3)} h_3^{(2)} h_4^{(3)} - h_1^{(1)} h_3^{(2)} h_3^{(3)} h_4^{(0)} - h_1^{(1)} h_3^{(2)} h_4^{(0)} h_4^{(3)} \\
& - h_1^{(2)} h_1^{(3)} h_2^{(0)} h_2^{(1)} - h_1^{(2)} h_1^{(3)} h_2^{(0)} h_3^{(1)} - h_1^{(2)} h_1^{(3)} h_2^{(0)} h_4^{(1)} - h_1^{(2)} h_1^{(3)} h_2^{(1)} h_3^{(0)} - h_1^{(2)} h_1^{(3)} h_2^{(1)} h_4^{(0)} \\
& - h_1^{(2)} h_1^{(3)} h_3^{(0)} h_3^{(1)} - h_1^{(2)} h_1^{(3)} h_3^{(0)} h_4^{(1)} - h_1^{(2)} h_1^{(3)} h_3^{(1)} h_4^{(0)} - h_1^{(2)} h_1^{(3)} h_4^{(0)} h_4^{(1)} - h_1^{(2)} h_2^{(0)} h_3^{(1)} h_3^{(3)} \\
& - h_1^{(2)} h_2^{(1)} h_2^{(3)} h_4^{(0)} - h_1^{(2)} h_2^{(1)} h_3^{(3)} h_4^{(0)} - h_1^{(2)} h_2^{(3)} h_4^{(0)} h_4^{(1)} - h_1^{(2)} h_3^{(0)} h_3^{(1)} h_3^{(3)} - h_1^{(2)} h_3^{(1)} h_3^{(3)} h_4^{(0)} \\
& - h_1^{(2)} h_3^{(3)} h_4^{(0)} h_4^{(1)} - h_1^{(3)} h_2^{(0)} h_2^{(1)} h_2^{(2)} - h_1^{(3)} h_2^{(0)} h_2^{(1)} h_4^{(2)} - h_1^{(3)} h_2^{(0)} h_2^{(2)} h_3^{(1)} - h_1^{(3)} h_2^{(0)} h_2^{(2)} h_4^{(1)} \\
& - h_1^{(3)} h_2^{(0)} h_3^{(1)} h_4^{(2)} - h_1^{(3)} h_2^{(1)} h_2^{(2)} h_3^{(0)} - h_1^{(3)} h_2^{(1)} h_2^{(2)} h_4^{(0)} - h_1^{(3)} h_2^{(1)} h_3^{(0)} h_3^{(2)} - h_1^{(3)} h_2^{(1)} h_3^{(0)} h_4^{(2)} \\
& - h_1^{(3)} h_2^{(1)} h_3^{(2)} h_4^{(0)} - h_1^{(3)} h_2^{(1)} h_4^{(0)} h_4^{(2)} - h_1^{(3)} h_2^{(2)} h_3^{(0)} h_3^{(1)} - h_1^{(3)} h_2^{(2)} h_3^{(0)} h_4^{(1)} - h_1^{(3)} h_2^{(2)} h_3^{(1)} h_4^{(0)} \\
& - h_1^{(3)} h_2^{(2)} h_4^{(0)} h_4^{(1)} - h_1^{(3)} h_3^{(0)} h_3^{(1)} h_3^{(2)} - h_1^{(3)} h_3^{(0)} h_3^{(1)} h_4^{(2)} - h_1^{(3)} h_3^{(0)} h_3^{(2)} h_4^{(1)} - h_1^{(3)} h_3^{(0)} h_4^{(1)} h_4^{(2)} \\
& - h_1^{(3)} h_3^{(1)} h_3^{(2)} h_4^{(0)} - h_1^{(3)} h_3^{(1)} h_4^{(0)} h_4^{(2)} - h_1^{(3)} h_3^{(2)} h_4^{(0)} h_4^{(1)} - h_1^{(3)} h_4^{(0)} h_4^{(1)} h_4^{(2)} - h_2^{(0)} h_2^{(1)} h_2^{(2)} h_2^{(3)} \\
& - h_2^{(0)} h_2^{(1)} h_2^{(2)} h_4^{(3)} - h_2^{(0)} h_2^{(2)} h_2^{(3)} h_3^{(1)} - h_2^{(0)} h_2^{(2)} h_2^{(3)} h_4^{(1)} - h_2^{(0)} h_2^{(2)} h_3^{(1)} h_3^{(3)} - h_2^{(0)} h_2^{(2)} h_3^{(1)} h_4^{(3)} \\
& - h_2^{(0)} h_2^{(2)} h_3^{(3)} h_4^{(1)} - h_2^{(0)} h_2^{(2)} h_4^{(1)} h_4^{(3)} - h_2^{(0)} h_2^{(3)} h_3^{(1)} h_4^{(2)} - h_2^{(0)} h_3^{(1)} h_3^{(3)} h_4^{(2)} - h_2^{(0)} h_3^{(1)} h_4^{(2)} h_4^{(3)} \\
& - h_2^{(1)} h_2^{(2)} h_2^{(3)} h_4^{(0)} - h_2^{(1)} h_2^{(2)} h_3^{(0)} h_4^{(3)} - h_2^{(1)} h_2^{(2)} h_4^{(0)} h_4^{(3)} - h_2^{(1)} h_2^{(3)} h_3^{(2)} h_4^{(0)} - h_2^{(1)} h_2^{(3)} h_4^{(0)} h_4^{(2)} \\
& - h_2^{(1)} h_3^{(2)} h_4^{(0)} h_4^{(3)} - h_2^{(2)} h_2^{(3)} h_3^{(1)} h_4^{(0)} - h_2^{(2)} h_2^{(3)} h_3^{(1)} h_4^{(1)} - h_2^{(2)} h_3^{(0)} h_3^{(1)} h_3^{(3)} - h_2^{(2)} h_3^{(0)} h_3^{(1)} h_4^{(3)} \\
& - h_2^{(2)} h_3^{(0)} h_4^{(1)} h_4^{(3)} - h_2^{(2)} h_3^{(1)} h_3^{(3)} h_4^{(0)} - h_2^{(2)} h_3^{(1)} h_4^{(0)} h_4^{(3)} - h_2^{(2)} h_3^{(3)} h_4^{(0)} h_4^{(1)} - h_2^{(2)} h_4^{(0)} h_4^{(1)} h_4^{(3)} \\
& - h_2^{(3)} h_3^{(1)} h_3^{(2)} h_4^{(0)} - h_2^{(3)} h_3^{(1)} h_4^{(0)} h_4^{(2)} - h_2^{(3)} h_3^{(2)} h_4^{(0)} h_4^{(1)} - h_2^{(3)} h_4^{(0)} h_4^{(1)} h_4^{(2)} - h_3^{(0)} h_3^{(1)} h_3^{(2)} h_3^{(3)} \\
& - h_3^{(0)} h_3^{(1)} h_3^{(3)} h_4^{(2)} - h_3^{(0)} h_3^{(1)} h_4^{(2)} h_4^{(3)} - h_3^{(1)} h_3^{(2)} h_3^{(3)} h_4^{(0)} - h_3^{(1)} h_3^{(2)} h_4^{(0)} h_4^{(3)} - h_3^{(1)} h_3^{(3)} h_4^{(0)} h_4^{(2)} \\
& - h_3^{(1)} h_4^{(0)} h_4^{(2)} h_4^{(3)} - h_3^{(2)} h_3^{(3)} h_4^{(0)} h_4^{(1)} - h_3^{(2)} h_4^{(0)} h_4^{(1)} h_4^{(3)} - h_3^{(3)} h_4^{(0)} h_4^{(1)} h_4^{(2)} - h_4^{(0)} h_4^{(1)} h_4^{(2)} h_4^{(3)}
\end{aligned}$$

and note that all 125 terms occur with a minus sign. Since the $h_i^{(j)}$ are all positive, this implies that $\beta D_4 < 0$, and since $D_4 > 0$, it follows that $\beta < 0$.

Final Note. If a single monomial term in any of the expansions for D_2 or D_4 or βD_4 in terms of the $h_i^{(j)}$'s had a negative sign, or if a single monomial term in any of the expansions for D_1 or D_3 had a single negative term it would be possible to choose a set of $h_i^{(j)}$'s so that at least one of the $(-1)^k D_k$'s or β would be positive (and hence that g_4 would also be positive), which would lead to a realization of the permutation set Q_0 .

INTELLIGENCE ADVANCED RESEARCH ACTIVITY (IARPA), OFFICE OF THE DIRECTOR OF NATIONAL INTELLIGENCE, WASHINGTON, D.C.

E-mail address: mark.i.heiligman@ugov.gov